Spin-It Faster: Quadrics Solve All Topology Optimization Problems That Depend Only On Mass Moments

Supplemental Material

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1 QUADRIC DISCRETIZATION

Our quadric discretization is based on a uniform-density triangulation (V, E, F) of the unit sphere $S^2 = \{x \in \mathbb{R}^3 : \|x\| = 1\}$, which is kept constant during optimization. In every iteration, the vertices of this discretization are mapped onto the quadric $Q = \{x \in \mathbb{R}^3 : \beta(x) = 0\}$ by the inverse Gauss map ν^{-1}. This means that a vertex $z \in V$ is mapped onto Q in such a way that the normal at $\nu^{-1}(z) \in Q$ is equal to z.

Assuming that the quadric is defined as the solution to the degree-two polynomial

$$\beta(x) = \langle x, Ax\rangle + \langle b, x\rangle + c,$$

the surface normal ν at a point $x \in Q$ is parallel to the gradient $\nabla\beta(x) = 2Ax + b$. To find the (unique) Euclidean point with a given oriented normal, compute

$$w = \sqrt{\frac{4\langle \nu, A^{-1}\nu\rangle}{\langle b, A^{-1}b\rangle - 4c}},$$

and set $x = A^{-1}(\nu/w - b/2)$. The case $w = 0$ indicates that ν is the normal vector of an ideal point. If the discriminant is negative, then there is no point, Euclidean or ideal, with ν as its normal vector. If Q is an ellipsoid, then we discretize it by interpreting every point $z \in V$ as a normal vector ν and mapping it onto Q through this computation. In this case, the discriminant will be positive for every point.

If Q is a hyperboloid, then the Gauss map is not surjective, i.e., certain points in V will not have a preimage under ν. In this case, the image of the Gauss map $\nu(Q)$ is a proper subset of S^2. A point $z \in V$ will be the normal of a Euclidean point if and only if $z \in \nu(Q)$. The boundary between of $\nu(Q)$ relative to S^2 is given exactly by the normal vectors of ideal points, which are characterized by the equation $0 = \langle \nu, A^{-1}\nu\rangle$.

Before we can map (V, E, F) back to Q, we need to truncate it at this boundary, and then only map the portion contained in $\nu(Q)$. We do this by finding all intersections of edges in E with the boundary of $\nu(Q)$. This amount to solving, for each $(z_1, z_2) \in E$, the quadratic equation

$$0 = \langle (1-t)z_1 + tz_2, A^{-1} \cdot ((1-t)z_1 + tz_2)\rangle$$

for $t \in (0, 1)$. If there is exactly one intersection, this implies that $z_1 \in \nu(Q)$ and $z_2 \notin \nu(Q)$ (or vice-versa), so the edge needs to be truncated. The new point $z_{\text{id}} = (1-t)z_1 + tz_2$ is the normal of an ideal point having homogeneous coordinates $(x_{\text{id}}, 0)$ with $x_{\text{id}} = A^{-1}z_{\text{id}}$. Once all edges have been processed, we stitch the vertices of $V \cap \nu(\Omega)$ to the newly created vertices by adding new triangles. Once mapped onto Q, these triangles will form the "semi-infinite" triangles by connecting Euclidean points to ideal points.

At this stage, we have obtained a triangulation of Q which contains both Euclidean and ideal points. In the last step, we will truncate this triangulation far away from the origin in order to remove the ideal vertices and be left with a closed surface. We do this so the resulting triangle mesh can be used as input to a mesh Boolean operation.

When a hyperboloid is intersected with a large sphere, the intersection will always consist of two closed loops which are approximately ellipsoidal in shape. We choose the truncation radius large enough to guarantee that this holds by recursively doubling the radius until the intersection contains two components. Then we add a closed triangle fan in each edge loop in order to close the hyperboloid on both sides. Note that the truncation radius is initialized larger than the diameter of Ω, so these triangle fans will not affect the result of the boolean operation.

2 NECESSARY KKT CONDITIONS

We formulate the relaxed problem (RP) from the main document in the standard form of an optimization problem on Banach spaces as follows: Let $X = L^\infty(\Omega)$, $Z = \mathbb{R}^k \times L^\infty(\Omega) \times L^\infty(\Omega)$. Define $G : X \to Z$ by $G(\chi) = (g(\chi), \chi, \chi - 1)$, and $K = \{0\}^k \times \{u \in X : u \geq 0\} \times \{v \in X : v \leq 0\} \subset Z$. Then, the problem (RP) can be written as

$$\min_{\chi \in X} f(x) \quad \text{s.t.} \quad G(\chi) \in K,$$

where K is closed and convex. The objective and equality constraint functions can be written as $f = \bar{f} \circ r$ and $g = \bar{g} \circ r$, where

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$\bar{f} : \mathbb{R}^n \to \mathbb{R}$ and $\bar{g} : \mathbb{R}^n \to \mathbb{R}^k$ are assumed continuously differentiable, and $r : X \to \mathbb{R}^n$ is the linear raw moment map. Thus f and g are Fréchet differentiable, with differentials $f'(\chi) \in L(X, \mathbb{R})$ and $g'(\chi) \in L(X, \mathbb{R}^k)$, where $L(U, V)$ denotes the space of bounded linear maps from U to V. To simplify the proof, we assume furthermore that $\bar{g}$ is affine, so it can be represented as $\bar{g}(x) = Ax + b$ with $A \in \mathbb{R}^{k\times n}$ and $b \in \mathbb{R}^k$. This holds in all applications shown in the main document; however, this is not strictly necessary for the KKT conditions to hold.

We need to verify a constraint qualification for the KKT conditions to hold; the regularity of the inequality constraints allows us to check the Abadie constraint qualification [Abadie 1965],

$$T_\ell(G, K, \chi) \subset T(F, \chi),$$

directly, where $F = \{x \in X : G(x) \in K\}$ denotes the feasible set. The linearizing cone at χ is defined as

$$T_\ell(G, K, \chi) = \left\{h \in X : G'(\chi)h \in \overline{\text{cone}(K - G(\chi))}\right\},$$

where cone(...) denotes the conical hull. The tangent cone simplifies to

$$T(F, \chi) = \overline{\text{cone}(F - \chi)}$$

because F is convex due to our simplifying assumption that g is affine.

Applying these definitions to $\chi^* \in F$, we find

$$\text{cone}(K - G(\chi^*)) = \{(0^k, u, v) \in Z : u(x) \geq 0 \text{ if } \chi^*(x) = 0, \\ v(x) \leq 0 \text{ if } \chi^*(x) = 1\},$$

which is closed, so

$$T_\ell(G, K, \chi^*) = \{h \in \ker g'(\chi^*) : h(x) \geq 0 \text{ if } \chi^*(x) = 0, \\ h(x) \leq 0 \text{ if } \chi^*(x) = 1\}.$$

The tangent cone can be written as

$$T(F, \chi^*) = \overline{\{\lambda(\varphi - \chi^*) : \lambda \geq 0, g(\varphi) = 0, 0 \leq \varphi \leq 1\}}.$$

Note that $\varphi - \chi^* \in \ker g'(\chi^*)$, because g is affine. Due to $0 \leq \varphi(x) \leq 1$, we have $\lambda(\varphi(x) - \chi^*(x)) \in [0, \infty)$ if $\chi^*(x) = 0$, and $\lambda(\varphi(x) - \chi^*(x)) \in (-\infty, 0]$ if $\chi^*(x) = 1$. For $\chi^*(x) \in (0, 1)$, we can pick $\lambda \geq 0$ and $\varphi(x)$ such that $\lambda(\varphi(x) - \chi^*(x))$ may attain any real number. By taking the closure, this lets us realize any $h \in T_\ell(G, K, \chi^*)$ within $T(F, \chi^*)$ as follows. If $h(x) = 0$, then $\varphi(x) = h(x)/\lambda + \chi^*(x) = \chi^*(x) \in [0, 1]$. If $h(x) > 0$, then $\chi^*(x) \in [0, 1)$. Therefore, $\varphi(x) = h(x)/\lambda + \chi^*(x) \geq 0$, and $\varphi(x) \leq 1$ for all $\lambda > 0$ great enough. If $h(x) < 0$, then $\chi^*(x) \in (0, 1]$. Therefore, $\varphi(x) = h(x)/\lambda + \chi^*(x) \leq 1$, and $\varphi(x) \geq 0$ for all $\lambda > 0$ great enough.

The KKT conditions guarantee the existence of a Lagrange multiplier $z \in \text{cone}(K - G(\chi^*))^\circ$, which is the polar cone

$$\text{cone}(K - G(\chi^*))^\circ = \{z \in Z^\star : \langle z, y\rangle \leq 0 \;\forall y \in \text{cone}(K - G(\chi^*))\},$$

where $(\cdot)^\star$ denotes the vector space dual. Using the expression of $\text{cone}(K - G(\chi^*))$ from above, we find that $z = (\lambda, \beta_0, \beta_1) \in \mathbb{R}^k \times L^\infty(\Omega)^\star \times L^\infty(\Omega)^\star$ such that $\beta_0(x) \leq 0$ if $\chi^*(x) = 0$, and $\beta_0(x) = 0$ otherwise. Likewise, $\beta_1(x) \geq 0$ if $\chi^*(x) = 0$, and $\beta_1(x) = 0$ otherwise.

The KKT equation in terms of z reads $f'(\chi^*) + \langle z, G'(\chi^*)\rangle = 0$. From $G'(\chi^*) = (\bar{g}'(r) \cdot r', \text{id}, \text{id})$, this expands to

$$(\bar{f}'(r) + \langle \lambda, \bar{g}'(r)\rangle) \cdot r'(\chi^*) + \beta_0 + \beta_1 = 0.$$

Because r is linear, the components of $r'(\chi^*)$ do not depend on χ^*, and are given by the constant, linear, and quadratic polynomials on Ω, i.e., $r'(\chi^*) = (1, \text{id}, \text{id} \otimes \text{id})$. To match the KKT conditions from Eq. 11 of the main document, set $\mathcal{L} = f + \langle \lambda, g\rangle$, and split up the product in the previous equation according to the constant, linear, and quadratic terms.

REFERENCES

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