Equivariant Gromov–Witten theory and GKM spaces

Daniel Holmes

Institute of Science and Technology Austria

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joint work with Giosuè Muratore

(CEMS.UL, University of Lisbon)

GKM Spaces

Algebraic setting:

Symplectic setting:

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- X …smooth projective variety / $\mathbb{C}$ with $\dim_{\mathbb{C}} X = n$
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Equivalently: The 1-skeleton

$$X_1 := \{x \in X : \dim(T \cdot x) \leq 1\}$$

is a finite union of T-stable S^2.

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Definition (continued)

The **axial function:**

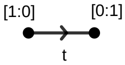
$$\alpha : E(G)^{\text{oriented}} \to \mathfrak{t}^*.$$

assigns to $e = (p, q)$ the T-weight of $T_p C_e$:

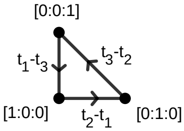
- $\alpha(\overline{e}) = -\alpha(e)$

Examples

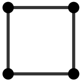
Toric varieties (smooth, projective)



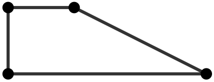
$\mathbb{CP}^1$



$\mathbb{CP}^2$



$\mathbb{CP}^1 \times \mathbb{CP}^1$



Hirzebruch surface

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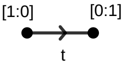
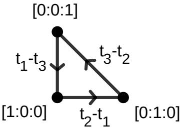
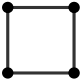
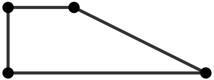


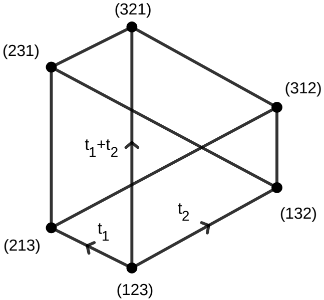






$\mathbb{CP}^1$ $\mathbb{CP}^2$ $\mathbb{CP}^1 \times \mathbb{CP}^1$ Hirzebruch surface

Homogeneous spaces [Guillemin–Holm–Zara '06]

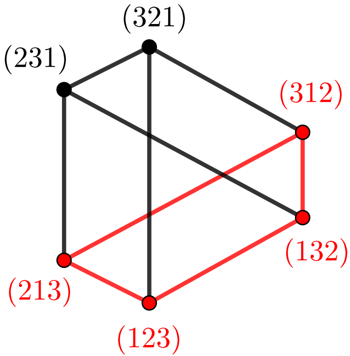


$\mathrm{Fl}(\mathbb{C}^3) \cong SU(3)/T$

vertices $\longleftrightarrow S_3$
edges $\longleftrightarrow (12), (23), (31)$

More Examples

Smooth Schubert varieties



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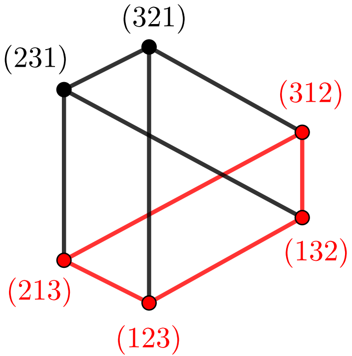

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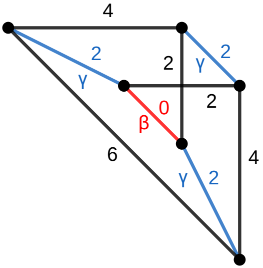
Twisted flag manifold

[Eschenburg '84, Tolman '98, Woodward '98, Goertsches–Konstantis–Zoller '19]



Not T-compatibly Kähler!

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- **Equivariant integration** [Atiyah–Bott '84, Berline–Vergne '83]

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$$\int_X y = \sum_{p \in V(G)} \frac{i_p^*(y)}{\prod_{e \in E(G)_p} \alpha(e)}$$

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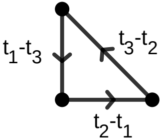
- **Equivariant cohomology** [Goresky–Kottwitz–MacPherson '98]

$$H_T^*(X;\mathbb{Q}) \cong \left\{ (f_p) \in \bigoplus_{p \in V(G)} \mathbb{Q}[t_1,\ldots,t_r] \;\middle|\; \begin{array}{c} \forall\, e = (p,q) \in E(G) \\ \alpha(e) \mid f_p - f_q \end{array} \right\}$$

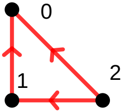
More from the GKM Graph

Betti numbers [Guillemin–Zara '01]

$\mathbb{CP}^2$:



$t_1 = 1$
$t_2 = 0$
$t_3 = 2$



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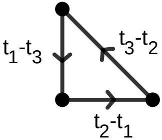

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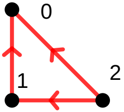

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Characteristic classes: for $p \in V(G)$, let $E(G)_p = \{\epsilon_1, \ldots, \epsilon_n\}$. Then

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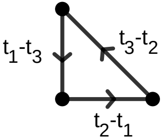

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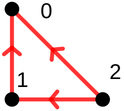

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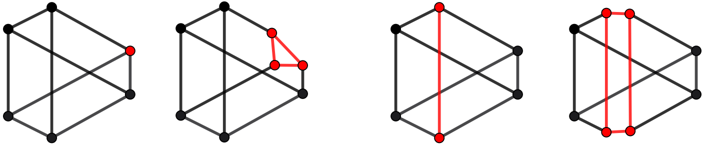
Blowups



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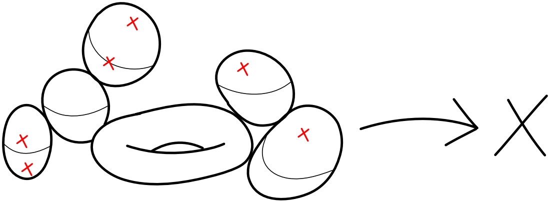
Moduli space of stable maps:

$$\overline{\mathcal{M}}_{g,m}(X;\beta) \xrightarrow{\mathrm{ev}} X^m$$



$$\mathrm{vdim}_{\mathbb{C}}\overline{\mathcal{M}}_{g,m}(X;\beta) = (1-g)(\dim_{\mathbb{C}} X - 3) + m + \int_\beta c_1(T_X)$$

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Example (Kontsevich)

$\#$ rational degree d curves in $\mathbb{CP}^2$ through $3d-1$ generic points:
$N_d := GW_{0,3d-1}^{\mathbb{CP}^2,d}([\mathrm{pt}], \dots, [\mathrm{pt}])$
$(N_d)_{d\geq 1} = (1, 1, 12, 620, 87304, \dots)$

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Λ . . . quantum coefficient ring, e.g. $\mathbb{Q}[[H_2^{\mathrm{eff}}(X)]]$

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Example

$QH^*(\mathbb{P}^n) = \mathbb{Q}[x]/(x^{n+1} - q)$

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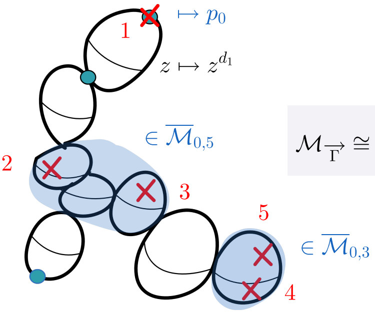
$\overrightarrow{s} : \{1, \ldots, m\} \to V(\Gamma)$

$\overrightarrow{g} : V(\Gamma) \to \mathbb{Z}_{\geq 0}$ with $\displaystyle |E(\Gamma)| - |V(\Gamma)| + 1 + \sum_{v \in V(\Gamma)} \overrightarrow{g}(v) = g$

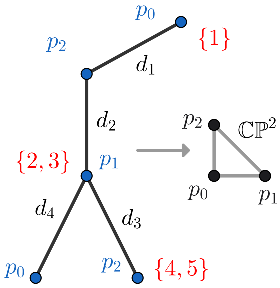
Localization on $\overline{\mathcal{M}}_{g,m}(X;\beta)$: Example

Components of $\overline{\mathcal{M}}_{0,m}(X;\beta)^T$ $\longleftrightarrow$ **Decorated trees $\vec{\Gamma}$**



Element of $\overline{\mathcal{M}}_{\vec{\Gamma}}$

$$\mathcal{M}_{\vec{\Gamma}} \cong \frac{\overrightarrow{\mathcal{M}}_{0,3} \times \overrightarrow{\mathcal{M}}_{0,5}}{\mathrm{Aut}}$$



$\vec{\Gamma}$

The Localization Formula

Theorem (Liu–Sheshmani '17, Hirschi '24)

Let X be an algebraic (resp. Hamiltonian) GKM space. Then

$$GW_{g,m}^{X,\beta}(y_1,\dots,y_m) = \sum_{\vec{\Gamma}} \underbrace{GW_{\vec{\Gamma}}(y_1,\dots,y_m)}_{\in\mathbb{Q}(t_1,\dots,t_r)}$$

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Here, $GW_{\vec{\Gamma}}$ is given in terms of

$$G \ , \ \alpha \ , \ \vec{\Gamma}$$

and, for each $e \in E(G)$, the splitting

$$\mathcal{N}_{C_e/X} \cong \mathcal{O}(a_1) \oplus \cdots \oplus \mathcal{O}(a_{n-1})$$

Curve classes

Proposition (H.–Muratore '25)

The GKM graph determines a full set of relations among

$$\{[C_e] : e \in E(G)\} \subset H_2(X;\mathbb{Z})$$

indexed by cycles generating $H_1(|G|)$.

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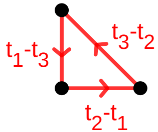

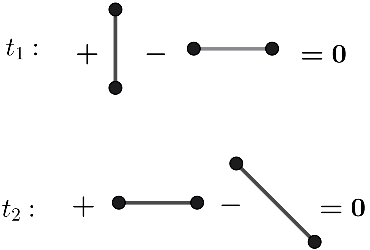

Independence of normal splittings

Lemma (H.–Muratore '25)

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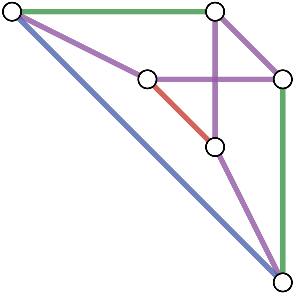
Consequence

The set of relevant $\overrightarrow{\Gamma}$ and their contributions are determined by (G, α) alone: equivariant GW invariants are computable combinatorially.

Tool: GKMtools.jl



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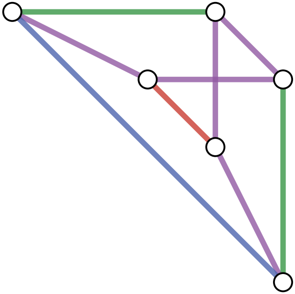

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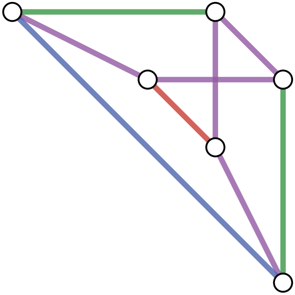

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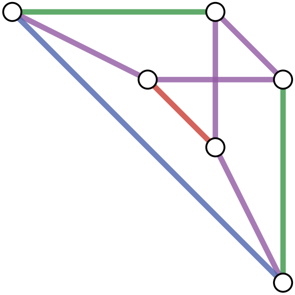

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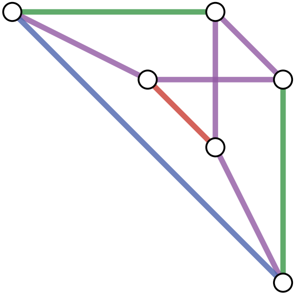

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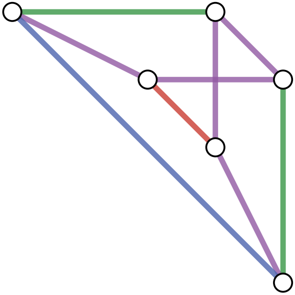

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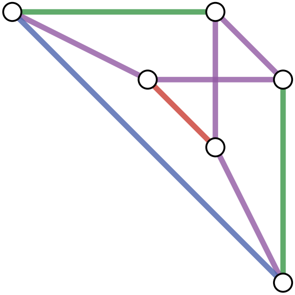

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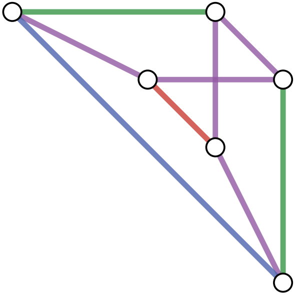

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- **Many important examples** (including Schubert varieties)

Almost Positive GKM Graphs

The **magnitude map** is

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X is *almost positive* if $\exists\, e_0 \in E(G)$ with

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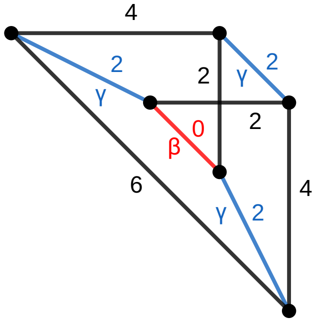
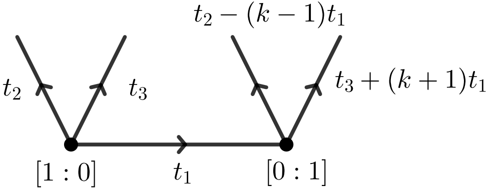
Example: the twisted flag manifold



The Local Model ($n = 3$)

Near e_0, (G, α) looks like

$$\begin{array}{c}\mathcal{O}(k-1) \oplus \mathcal{O}(-k-1) \\ \downarrow \\ C_{e_0}\end{array}$$



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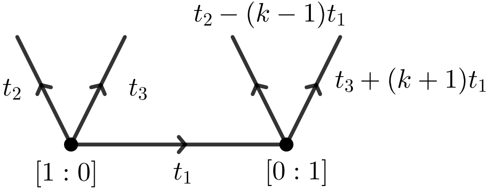

Theorem (H.–Muratore '25)

For any almost positive Hamiltonian GKM space X with $\dim_{\mathbb{R}} X = 6$,

(1) $k = 0$, *or*

(2) $k = 1$ *and* $t_3 = -t_1 + yt_2$ *for some* $y \in \mathbb{Z} \setminus \{0\}$, *or*

(3) $k \geq 1$ *and* $t_3 = -kt_1 + t_2$, *or*

(4) $t_1 + t_2 + t_3 = 0$

Quantum Products ($n = 3$)

Theorem (H.–Muratore '25 continued)

For any $y_1, y_2 \in H_T^*(X)$, we have

$$y_1 *_T y_2 = y_1 \smile y_2 + \mathrm{PD}(C_{e_0}) \left(\int_{C_{e_0}}^T y_1\right) \left(\int_{C_{e_0}}^T y_2\right) \cdot (\dagger)$$
$$+ \left(q^\gamma \text{ terms with } \int_\gamma c_1(T_X) > 0\right)$$

where $(\dagger)$ is given in terms of $\mathbf{q} := q^{[C_{e_0}]}$ by

(1) $\dfrac{\mathbf{q}}{1-\mathbf{q}}$

(2) $\dfrac{y\mathbf{q}}{1-\mathbf{q}}$

(3) $\dfrac{\mathbf{q}}{1-\mathbf{q}}$

(4) $\displaystyle\sum_{d>0} \frac{(-1)^{d(k+1)+1}}{k^2} \binom{k^2 d}{d} \mathbf{q}^d$

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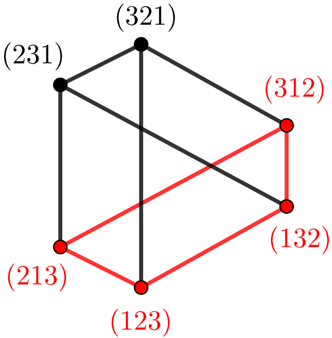

A Finiteness Criterion

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Example

G/B and G/P always satisfy this. But what about $X_w \subsetneq G/P$?

Strategy: GKM Theory for Schubert Varieties

GKM graph of G/P:
[Guillemin–Holm–Zara '06]

$V = W_G/W_P$

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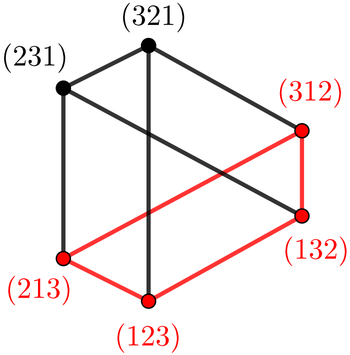
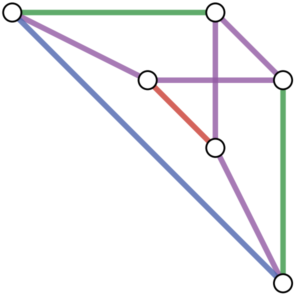
Example: $X_{(312)} \subset SL_3/B$



Example application of GKMtools.jl



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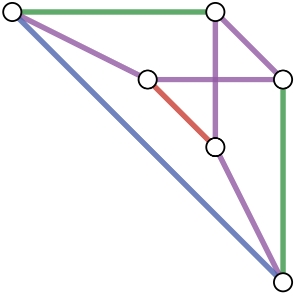

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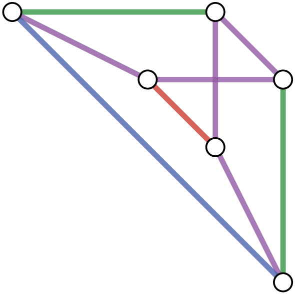

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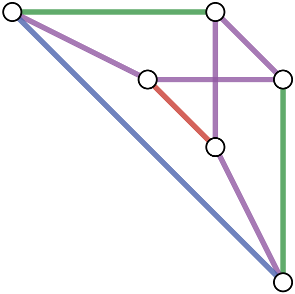

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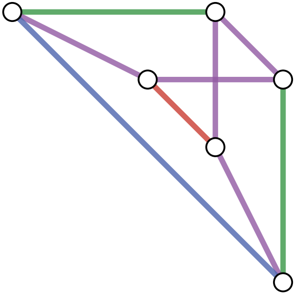

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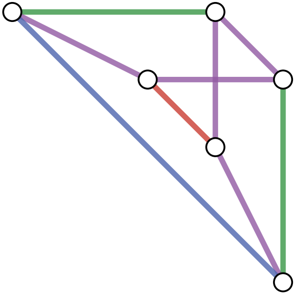

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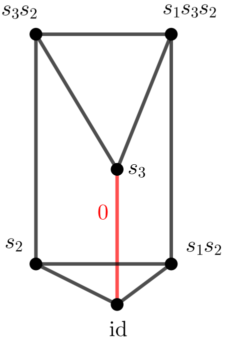
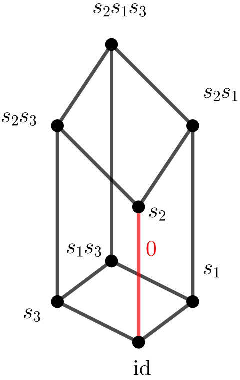
    - Flag almost positive X_w

Results

Two type A_3 smooth Schubert varieties which are almost positive



$X_w \subset \mathrm{Fl}_{2,1,1}(\mathbb{C}^4)$



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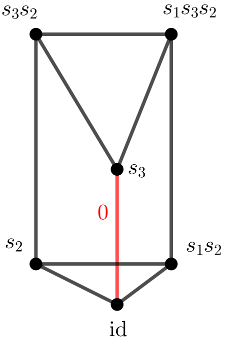

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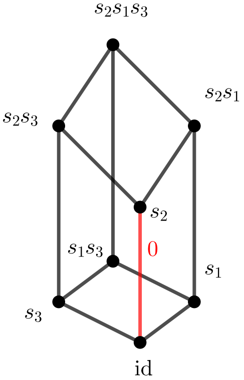

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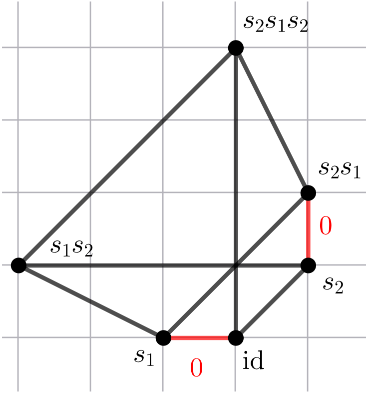
Theorem (H.–Muratore '25)

$\exists$ smooth Schubert varieties with infinite quantum products.

Results

An interesting example in type G_2



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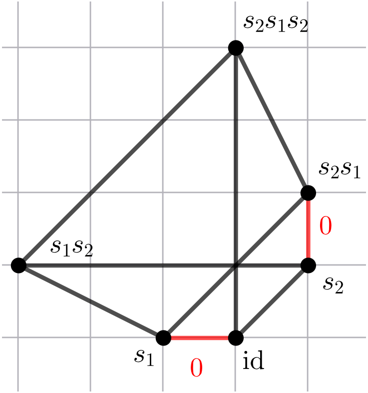

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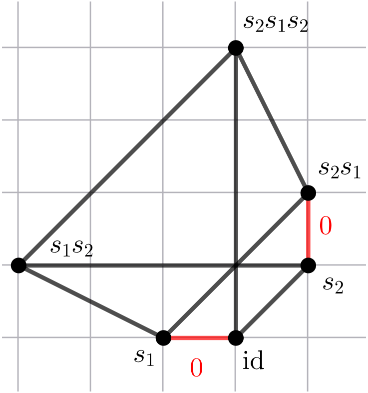

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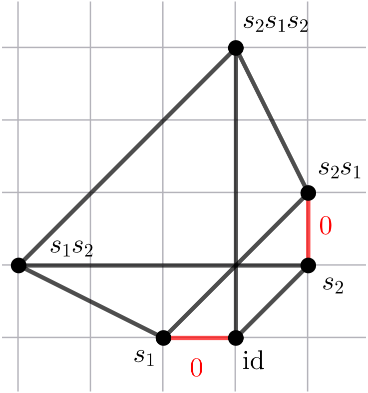

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Thank you for listening!

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