

Unless stated otherwise, let M be a smooth manifold.

Exercise 1. Let $M = \mathbb{R}^2$ with coordinates x, y and let $f : M \rightarrow \mathbb{R}$ be given by

$$f(x, y) := x^3 - 3y \cos(x) + y^2.$$

Calculate

$$df \left(\frac{\partial}{\partial x} \right) \quad \text{and} \quad df \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right)$$

as elements of $C^\infty(M)$.

Exercise 2. Let $\alpha \in \Omega^k(M)$, $p \in M$, and $v_1, \dots, v_k \in T_p M$ be linearly dependent¹. Prove that $\alpha_p(v_1, \dots, v_k) = 0$ and deduce that $\Omega^k(M) = 0$ for $k > \dim(M)$.

Exercise 3. Let $M = \mathbb{R}^3$ with coordinates x, y, z , and let $\alpha = dx + x^2 dy \in \Omega^1(M)$ and $\beta = dy \wedge dz \in \Omega^2(M)$.

- (i) Calculate $\alpha \wedge \beta$ and $(\alpha \wedge \beta)(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z})$.
- (ii) For vector fields v_1, v_2, v_3 on M , give an interpretation of the cases that $(\alpha \wedge \beta)(v_1, v_2, v_3)$ is positive, negative, or zero at a point p .

Exercise 4. Prove at least one of the following properties of the wedge product.

- (i) $\alpha \wedge \beta = (-1)^{a \cdot b} \beta \wedge \alpha$ where $\alpha \in \Omega^a(M)$ and $\beta \in \Omega^b(M)$.
(Does this mean that $\beta \wedge \beta = 0$ for any differential form β on M ?)
- (ii) $(\alpha \wedge \beta) \wedge \gamma = \alpha \wedge (\beta \wedge \gamma)$ for differential forms α, β, γ on M .
- (iii) Let $f : M \rightarrow N$ be a smooth map between manifolds M and N , and let α, β be differential forms on N . Then $f^*(\alpha \wedge \beta) = f^*(\alpha) \wedge f^*(\beta)$.

Exercise 5. Let $M = S^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$ be the unit sphere with north and south pole $P_\pm = (0, 0, \pm 1)$.

- (i) Consider the cylinder $C := \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = 1, |z| \leq 1\}$ with local² coordinates (θ, z) corresponding to $(\cos \theta, \sin \theta, z)$. Let $\pi : S^2 \setminus \{P_+, P_-\} \rightarrow C$ be the projection onto the cylinder preserving the z -coordinate³. Calculate $\pi^*(d\theta \wedge dz)$ in terms of dx, dy, dz .
- (ii) Write down a 2-form α on S^2 that is closed but not exact.

Exercise 6. Prove at least one of the following properties of the exterior derivative d .

- (i) Let $\omega \in \Omega^{\dim(M)}(M)$. Then $d\omega = 0$.
- (ii) Let $f : M \rightarrow N$ be a smooth map between manifolds M and N , and let α be a differential form on N . Then $d(f^*\alpha) = f^*(d\alpha)$.
- (ii) $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^a \alpha \wedge d\beta$ where $\alpha \in \Omega^a(M)$ (Leibniz rule)

¹i.e., there exist $a_1, \dots, a_k \in \mathbb{R}$ not all zero such that $a_1 v_1 + \dots + a_k v_k = 0$.

²Recall that the angle coordinate is only well-defined locally.

³More precisely, given a point $p \in S^2 \setminus \{P_+, P_-\}$, we obtain $\pi(p)$ by first projecting p orthogonally onto the z -axis and then intersecting the ray from that point on the z -axis through p with C .