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REPORTING INTERSECTIONS OF GEOMETRIC OBJECTS

BY MEANS OF COVERING RECTANGLES

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1. Introduction.

Finding all intersecting pairs of a set of objects in the plane (or in higher dimensions) is an important problem in the study of algorithmic geometry. The significance of solutions for this problem stems, firstly, from their applicability to more complicated geometric tasks involving the above as a subproblem, and secondly, from the tendency of applying them to related but non-geometric issues in Computer Science.

In the last two years, a series of papers appeared focussing on the intersection of 'orthogonal' objects. Edelsbrunner [2] and McCreight [5] independently solved the problem involving n axis-parallel rectangles in the plane in time O(nlogn+t) and space O(n), where t intersecting pairs are reported. All intersecting pairs of a set of d-dimensional axis-parallel boxes can be reported in time $O(n\log^{d-1}n+t)$ and space $O(n\log^{d-2}n)$, consult Edelsbrunner [3] and Edelsbrunner, Maurer [4].

In this paper, we consider more general geometric objects, so-called regions. A practical algorithm based on the idea of covering arbitrary regions by axis-parallel rectangles, resp. axis-parallel d-dimensional boxes, is investigated. It is shown to be more efficient even in the worst case than thought before, provided the set of regions satisfies a weak condition: The regions must not be 'too thin' and of 'too diverse' size.

2. Intersecting Regions in the Plane.

A region in the plane is a subset of the plane, such that there exist (i) a largest non-empty axis-parallel square entirely contained in the region, (ii) a smallest bounded axis-parallel square covering the region, and (iii) an algorithm which determines in constant time the smallest covering axis-parallel rectangle for the region. In addition, we demand that whether two given regions intersect or not can be decided in constant time.

Let r(A) and R(A) denote the radii of the above mentioned squares. (The radius of a square is one half of the length of one side.) Note that r(A) is strictly positive and that R(A) is well defined, i.e. R(A) is a finite real.

The problem can now be precisely formulated: Let S be a set of n regions in the plane and let r(S), resp. R(S), denote the minimum of r(A), resp. the maximum of R(A), over all regions A in S. We confine attention to sets S for which the ration R(A)/r(A) is bounded above by a constant. The task is to report all intersecting pairs of regions in S.

We suggest the following simple solution which is of practical and of theoretical interest as will be demonstrated later:

Algorithm:

Initially, compute for each region its smallest covering rectangle. (All occurring rectangles and squares are assumed to be axis-parallel. For simplification, this will not be noted anymore.) Then use one of the algorithms mentioned above to determine all intersecting pairs of rectangles. Test for each intersecting pair whether the associated regions intersect. If they do then report the pair.

Let t denote the number of intersecting pairs of regions in S and let t' denote the number of intersecting pairs of the covering rectangles. From straightforward observations, one can derive that the above algorithm runs in time O(nlogn+t') and requires O(n) space.

In what now follows, t' will be shown to be proportional to t, unless t' is very small, i.e. t' is at most linear in n. (The basic idea of the proof originates from Bentley, Stanat, Williams [1] who investigated algorithms to report all 'near neighbor pairs' of a given set of points.)

Theorem: Let S be a set of n regions as described in the formulation of the problem above. Then $t' = \Theta(t) + O(n)$.

Proof: Associate with each region A of S a point p(A) in its interior, such that the square with radius r(S) centered at p(A) is entirely contained in A. Note that the square with radius 2R(S) centered at p(A) entirely covers A, and that 2R(S)/r(S) is equal to a constant C independent of S.

We will consider t_1, the number of intersections among the small squares centered at the points p(A), for all A in S, and t_2, the number of intersections among the large squares centered at the points p(A). Obviously, $t_1 \leq t \leq t' \leq t_2$. The proof would be finished, could we show that t_2 is proportional to t_1, unless t_2 is at most linear in n.

Consider an axis-parallel grid which imposes a decomposition of the plane into disjoint squares with radii r(S) which we henceforth call cells. Each cell Z_i is associated with $|Z_i|$, the number of points p(A) that lie in it. Hence, $\frac{1}{2}\sum_i |Z_i|(|Z_i|-1)$ is a lower bound for t_1.

Since 2R(S)/r(S) is equal to C, two regions A and B may intersect only if their associated points p(A) and p(B) lie in cells 'not too far' from each other. For each cell Z_i we define its neighborhood N_i as the set of cells Z_j which intersect the square with radius 4R(S)+ r(S) centered at the center of Z_i. Note that N_i contains exactly $(2\lceil C\rceil+1)^2$ cells including Z_i itself. The following calculation yields an upper bound for t_2:

$$\begin{aligned}
\frac{1}{2}\sum_i |Z_i|\Big(\sum_{Z_j \text{ in } N_i} |Z_j|\Big) &= \frac{1}{2}\sum_i \sum_{Z_j \text{ in } N_i} |Z_i||Z_j| \leq \\
&\leq \frac{1}{2}\sum_i \sum_{Z_j \text{ in } N_i} \frac{1}{2}(|Z_i|^2+|Z_j|^2) = \\
&= \frac{1}{4}\Big(\sum_i \sum_{Z_j \text{ in } N_i} |Z_i|^2 + \sum_i \sum_{Z_j \text{ in } N_i} |Z_j|^2\Big) \leq \\
&\leq \frac{1}{2}(2\lceil C\rceil+1)^2 \sum_i |Z_i|^2.
\end{aligned}$$

This implies that t_2 is proportional to t_1, unless t_2 is at most linear in n which concludes the proof.

An immediate consequence of the theorem with regard to the complexity of our algorithm is formulated in a corollar.

Corollar: Let S be a set of n regions in the plane such that the ratio $R(S)/r(S)$ is bounded above by a constant. Then all t intersecting pairs of regions can be reported in time $O(n\log n+t)$ and space $O(n)$.

3. Discussion.

The near neighbor problem asks for all pairs of points of a given finite set no more than a fixed distance apart. Using the Euclidean metric as distance function, this problem can be solved by finding all intersecting pairs of a finite set of equalsized circles. Thus, our result implies that all t Euclidean 'near neighbor pairs' of a set of n points can be reported in time $O(n\log n+t)$ even in the worst case. In addition, our algorithm does not rely on the Euclidean metric. The same result can be obtained for any metric for which the following conditions hold: (i) The distance between two points can be computed in constant time, (ii) there exists a non-empty square which is entirely contained in the union circle defined by the metric at hand, and (iii) there exists a bounded square containing the union circle of the metric.

It seems worthwhile to mention that our algorithm works with sets of regions with considerably weakened restrictions as well. We need not insist on the possibility to determine the smallest covering rectangle of a region in constant time. Instead, all smallest covering rectangles must be computable in time $O(n\log n)$. Although the replacement of the regions by their smallest covering rectangle minimizes the number of explicit intersection tests, the proof of the above theorem implies that larger rectangles work with the same asymptotical complexity.

Last not least, it has to be stated that the above observations are not restricted to two dimensions. From straightforward generalizations one can derive analogous algorithms that work in three and more dimensions based on preliminary results of Edelsbrunner [3] and Edelsbrunner, Maurer [4].

References.

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