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On the Number of Line Separations of a Finite Set in the Plane

H. EDELSBRUNNER AND E. WELZL

Institutes for Information Processing, Technical University of Graz, Schiesstattgasse 4a, A-8010 Graz, Austria

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Let S denote a set of n points in the Euclidean plane. A subset S' of S is termed a k-set of S if it contains k points and there exists a straight line which has no point of S on it and separates S' from $S - S'$. We let $f_k(n)$ denote the maximum number of k-sets which can be realized by a set of n points. This paper studies the asymptotic behaviour of $f_k(n)$ as this function has applications to a number of problems in computational geometry. A lower and an upper bound on $f_k(n)$ is established. Both are nontrivial and improve bounds known before. In particular, $f_k(n) = f_{n-k}(n) = \Omega(n \log k)$ is shown by exhibiting special point-sets which realize that many k-sets. In addition, $f_k(n) = f_{n-k}(n) = O(nk^{1/2})$ is proved by the study of a combinatorial problem which is of interest in its own right. © 1985 Academic Press Inc.

1. INTRODUCTION

Let k and n denote two positive integers such that $k \leqslant n - 1$, and let S denote a set of n points in the Euclidean plane. We call a subset S' of S a *k-set of S* if S' contains k points and there exists a straight line with no point of S on it which separates S' from $S' - S$. The number of k-sets realized by S is denoted by $f_k(S)$. While it is well known that

$$\sum_{k=1}^{n-1} f_k(S) \leqslant n^2 - n$$

which follows from a result in Steiner [14], little is known about the individual terms $f_k(S)$. For positive integers k and n, $k \leqslant n - 1$, we let $f_k(n)$ denote the maximum of $f_k(S)$ for all sets S of n points. The technical sections will establish a lower and an upper bound on the asymptotic behaviour of $f_k(n)$. To the knowledge of the authors, the problem was not investigated before. (The following notation is used for the description of the asymptotic behaviour: Let $f(n)$ and $g(n)$ denote two positive-valued

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functions. We say $f(n)=O(g(n))$ if there exists a positive constant c such that $f(n) \leqslant cg(n)$, for all positive integers n. In addition, we say $f(n)=\Omega(g(n))$ if $g(n)=O(f(n))$, and $f(n)=\Theta(g(n))$ if $f(n)=O(g(n))$ and $f(n)=\Omega(g(n))$.)

The motivation for our interest in $f_k(n)$ stems from several applications to problems in computational geometry. This young field of computer science studies the computational complexity of elementary geometric constructions, cf. Shamos [12] for an excellent introduction to the field.

One typical problem for which knowledge about $f_k(n)$ is important is called the *halfplanar range search problem*. It requires the accommodation of a finite set of points in the Euclidean plane in a data structure such that the number of points which lie in a later specified query halfplane can be determined efficiently. Such a query is called a *halfplanar range query*.

Two nontrivial solutions for this problem are described in Willard [15] and in Edelsbrunner, Kirkpatrick, and Maurer [2]. These solutions have the disadvantage of either requiring a rather large amount of time for answering a query ($O(n^{0.77})$ time in [15]) or requiring a rather large amount of space for the data structure ($O(n^2)$ space in [2]). The dynamic setting of the halfplanar range search problem asks for the computation of halfplanar range queries, insertions of new points into the data structure, and deletions of old points from the data structure. For this setting, Fredman [5] provides evidence that for any solution of the problem, $\Omega(n^{1/3})$ time is required by the most time consuming of the three operations. This substantiates the thesis that the problem is inherently more difficult than the classical orthogonal range search problem, see, e.g., Bentley and Friedman [1].

By virtue of these disappointing facts, Edelsbrunner and Welzl [3] investigated the possibility of providing approximate solutions for the halfplanar range search problem. The simplest version asks whether or not there are at least half of the n given points in the query halfplane. For this problem a data structure is developed in [3] which requires $O(f_{n/2}(n))$ space and permits us to answer a query in $O(\log n)$ time.

A second problem related to $f_k(n)$ again deals with a set S of n points in the Euclidean plane. S is to be accommodated in a data structure such that the k of those points nearest to a later specified query point can be determined efficiently. The parameter k is a fixed positive integer smaller than n. This so-called *k-nearest neighbours search problem* was solved by Shamos and Hoey [13] and Lee [8] using order-k Voronoi diagrams. The order-k Voronoi diagram for S consists of a collection of nonoverlapping regions which covers the plane. Each subset of k points of S is assigned a (potentially empty) region R such that those k points are the k nearest to a query point q if and only if q resides in R.

It is easily verified that the region assigned to k specific points is properly unbounded (i.e., there are two bounding rays which are not parallel) if and only if those points define a k-set of S. Thus $f_k(S)$ denotes the number of properly unbounded regions of the diagram and $f_k(n)$ denotes the maximal number of properly unbounded regions which can occur in any order-k Voronoi diagram for n points.

Finally, $f_k(n)$ applies to a problem which deals with a set S of n points on a horizontal line. Each one of those points moves with constant but, in general, unique speed towards the left or the right. Ottmann and Wood [10] studied a number of questions concerned with sets of moving points on the line.

We say a point *is at position* k (at some point in time) if it is the only kth point from the right. In general, the points at position k change over time and a single point is, in general, at position k during several intervals of time.

THEOREM 1.1. *Let S be a set of n moving points on the line and let $L_k(S)$ denote the sequence of points at position k, considering the time interval from minus to plus infinity. Then the length of $L_k(S)$ is no greater than $f_{k-1}(n) + f_k(n) + 1$.*

Proof. Let each point p of S be specified by its location at time 0 and by its speed which is positive if it moves towards the right and negative if it moves towards the left. Let S' be a set of points in the plane such that $p' = (p_1, p_2)$ is in S' if and only if there is a moving point p in S with location p_1 at time 0 and speed p_2. The rightmost k points of S at time t correspond to the k points in S' which lie to the right of a line with slope $-1/t$ which, in turn, separates those k points from the rest. (The verification of this assertion is left to the interested reader.) The point at position k changes when the current kth point changes its position with the $(k-1)$th or $(k+1)$th point. Thus, each change of the point at position k defines a new $(k-1)$-set or k-set of S'. The easily verified observation that no $(k-1)$-set or k-set occurs twice during this process completes the argument.

An efficient algorithm which computes the sequence of points at some fixed position k can be found in Edelsbrunner and Welzl [3].

The organization of the paper is as follows: Section 2 presents several basic observations on the behaviour of $f_k(n)$. Then Section 3 demonstrates a lower bound on $f_k(n)$ by exhibiting special sets of planar points. Section 4 establishes an upper bound on $f_k(n)$ by investigating a combinatorial problem which is also of interest in its own right. Finally, Section 5 discusses the contributions and gives some related open problems.

2. BASIC OBSERVATIONS

This section is intended to provide the reader with an appropriate feeling for the behaviour of $f_k(n)$.

Observation 2.1. $f_k(n) = f_{n-k}(n)$ for $1 \leqslant k \leqslant n-1$.

Thus, the interesting values of k are those from 1 up to $\lfloor n/2 \rfloor$. (For any real number a, $\lfloor a \rfloor$ denotes the largest integer not greater than a. Similarly $\lceil a \rceil = -\lfloor -a \rfloor$.) It is also trivial to evaluate $f_k(n)$ for $k = 1$.

Observation 2.2. $f_1(n) = n$ for $n \geqslant 2$.

$f_1(n)$ is realized by a set of n points which all are extreme points of the set. Less trivial is the evaluation for $k = 2$.

LEMMA 2.3. *$f_2(n) = \lfloor 3n/2 \rfloor$ for $n \geqslant 4$.*

Proof. We show the existence of a point-set S which realizes that many 2-sets. For $n = 4$, the set S contains three points which define a nondegenerate triangle and a fourth point in the interior of this triangle. For $n > 4$, take a regular $\lceil n/2 \rceil$-gon and choose the remaining $\lfloor n/2 \rfloor$ points just in from the midpoints of all (or all but one) of the sides, as indicated in Fig. 2.1.

The following counting argument implies the lower bound of the assertion: Any two consecutive points of the first chosen regular $\lceil n/2 \rceil$-gon define a 2-set and each of the additional points defines two 2-sets with the two nearest points from the first chosen $\lceil n/2 \rceil$-gon, respectively. This gives $\lceil n/2 \rceil + 2\lfloor n/2 \rfloor = \lfloor 3n/2 \rfloor$ 2-sets.

We leave the proof that $\lfloor 3n/2 \rfloor$ is also an upper bound to the interested reader. We only note that this proof becomes easy in the setting of Section 4. This completes the argument.

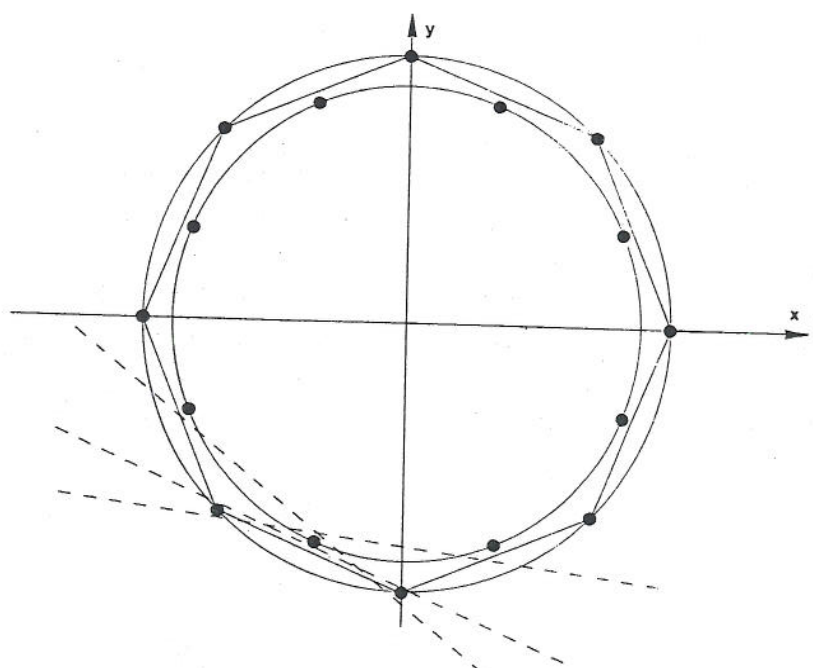

FIG. 2.1. 16 points which realize 24 2-sets.

Intuitively, it seems to be clear that for calculating $f_k(n)$ we can restrict our attention to sets of points in general position, that is, no three points are collinear, and no lines through two points, respectively, are parallel. Let us be more precise:

LEMMA 2.4. *Let k and n denote two positive integers with $k < n$. Then there exists a set S of n points in general position such that $f_k(S) = f_k(n)$.*

Proof. If S_0 is a set of n points such that $f_k(S_0) = f_k(n)$, each k-set T and its corresponding $(n-k)$-set T' are determined by some line $L(\{T, T'\})$. The arrangement of these lines $L(\{T, T'\})$ partitions the plane into open regions. Each point of S_0 can be moved within its region to give a new set S in general position with the same k-sets as S_0. This completes the argument.

3. A Lower Bound on the Maximal Number of k-Sets

This section demonstrates a lower bound on $f_k(n)$. This is done by exhibiting special point-sets and analyzing the number of k-sets they realize. We start with describing several sets of n points in the plane and analyzing the number of $\lfloor n/2 \rfloor$-sets they realize. Step by step, the point-set is refined which improves the result. Finally, the lower bound is generalized to arbitrary k with $1 \leqslant k \leqslant n-1$.

Let n be a multiple of 6 and let $S_1(n)$ denote a set of n points in the plane as follows: r, s, and t are three rays emanating from the origin which contain the points of $S_1(n)$. Any two of r, s, and t enclose an angle of $2\pi/3$, see Fig. 3.1. We choose $n/3$ points of $S_1(n)$ on r, s, and t, respectively, such that none of these points coincides with the origin. The three subsets are called S_1^r, S_1^s, and S_1^t, see Fig. 3.1.

Calculation 3.1. Let n be a multiple of 6 and let $S_1(n)$ be as described above. Then $f_{n/2}(S_1(n)) = n + 6$.

Proof. The assertion follows from an easy counting argument to be presented: For each positive i with $1 \leqslant i \leqslant n/6 - 1$ there is a unique $n/2$-set which contains all $n/3$ points of S_1^r, the i points of S_1^s which are nearest to the origin, and the $n/6 - i$ points of S_1^t nearest to the origin. The complement of each one of those $n/6 - 1$ $n/2$-sets is again an $n/2$-set. Since the same argument holds when S_1^r is replaced by S_1^s or S_1^t we now have identified $n - 6$ $n/2$-sets of $S_1(n)$. Note that none of these $n/2$-sets contains exactly $n/6$ points of any one of S_1^r, S_1^s, and S_1^t, see Fig. 3.1.

There exist exactly two $n/6$-sets of S_1^r which both can be combined with S_1^s or S_1^t, see Fig. 3.1. The same reasoning holds for S_1^s and S_1^t replacing S_1^r. This gives additional 12 $n/2$-sets of $S_1(n)$ which completes the argument.

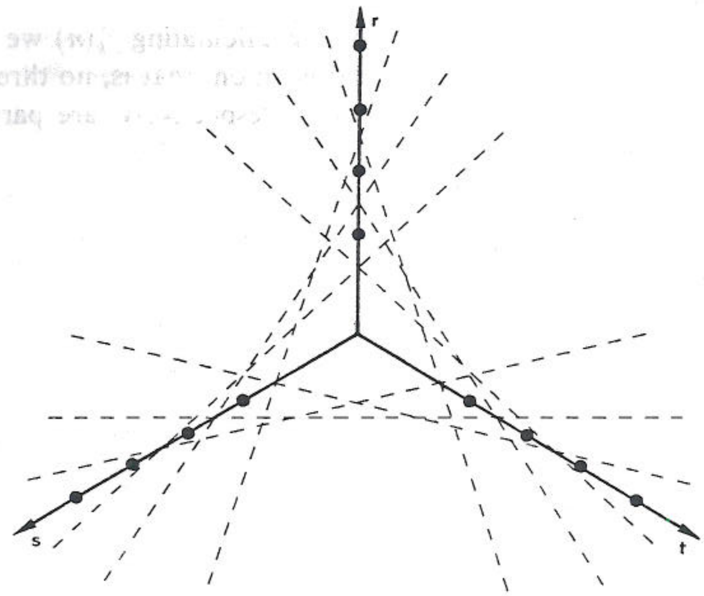

FIG. 3.1. A possible $S_1(12)$ with $f_6(S_1(12)) = 18$.

A short moment of reflection shows that the special configuration chosen for $S_1(n)$ allows many $n/2$-sets which do not partition one of S_1^r, S_1^s, or S_1^t into two equal sized halves. On the other hand, the fact that the points are chosen exactly on the rays keeps the number of remaining $n/2$-sets small. We will now describe a point-set $S_2(n)$ which shares with $S_1(n)$ the kind of configuration but does not share the mentioned shortcoming.

Let n again be a multiple of 6 and let S_2^r, S_2^s, and S_2^t contain $n/3$ points each such that $S_2(n) = S_2^r \cup S_2^s \cup S_2^t$. Intuitively, we obtain S_2^A from S_1^A for $A = r, s, t$ by perturbing a bit the points in the latter set.

The points of S_2^r, S_2^s, and S_2^t are chosen near r, s, and t and far enough from the origin, respectively. To be precise we choose two positive real numbers ε and δ and define three regions r', s', and t' consisting of those points of the plane whose distance to r, s, and t is smaller than ε, respectively, and whose orthogonal projections onto r, s, and t, respectively, are further from the origin than δ units of length, see Fig. 3.2. ε is chosen sufficiently small and δ is assumed to be sufficiently large. The points of S_2^r, S_2^s, and S_2^t are chosen in r', s', and t', respectively, such that for any permutation (A, B, C) of (r, s, t)

(i) the points of S_2^A are in general position, and

(ii) the line through each pair of points of S_2^A separates B' from C'.

Point-sets which satisfy condition (i) exist which can be shown by an easy inductive argument. Condition (ii) can be guaranteed by moving the points of S_2^A for $A = r, s, t$ sufficiently near to A.

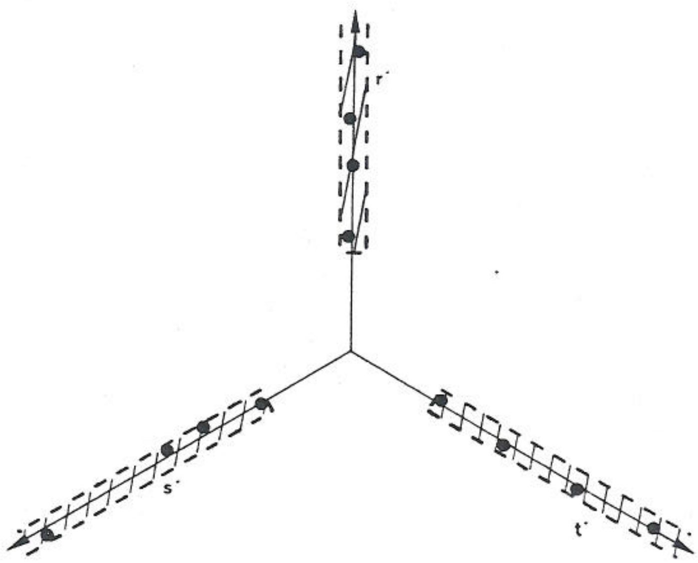

FIG. 3.2. A possible $S_2(12)$.

Calculation 3.2. Let n be a multiple of 6 and let $S_2(n)$ be a point-set as described above. Then $f_{n/2}(S_2(n)) = n + f_{n/6}(S_2^r) + f_{n/6}(S_2^s) + f_{n/6}(S_2^t)$.

Proof. Due to condition (ii) there are $n - 6$ $n/2$-sets of $S_2(n)$ which do not contain exactly half of the points of one of S_2^r, S_2^s, and S_2^t.

Let us now consider the $n/2$-sets of $S_2(n)$ which contain exactly $n/6$ points of S_2^r, say. We first concentrate on those $n/6$-sets of S_2^r which neither contain the $n/6$ points nearest to the origin nor those $n/6$ points farthest from the origin. A line L which separates two of those $n/6$-sets of S_2^r also separates s' from t' and therefore also separates two $n/2$-sets of $S_2(n)$. Applying the same argument to S_2^s and S_2^t yields $f_{n/6}(S_2^r) + f_{n/6}(S_2^s) + f_{n/6}(S_2^t) - 6$ $n/2$-sets of $S_2(n)$.

The $n/6$ points of S_2^r nearest to or farthest from the origin can be combined with S_2^s and with S_2^t. Replacing S_2^r once by S_2^s and once by S_2^t yields 12 $n/6$-sets of $S_2(n)$ of the latter kind. Adding the three numbers derived completes the argument.

For the next refinement of the point-set we need a transformation which does not change the number of k-sets. To this end let $y(\varepsilon, \delta)$ denote the region consisting of those points above the x axis whose distance to the y axis is smaller than ε and whose distance from the x axis is larger than δ and smaller than 2δ for positive real numbers ε and δ. We also define two points $x' = (-\varepsilon, 0)$ and $x'' = (\varepsilon, 0)$, see Fig. 3.3.

LEMMA 3.3. *Let S denote a set of m points in general position. Then for any two positive real numbers ε and δ there is a set S' of points in $y(\varepsilon, \delta)$*

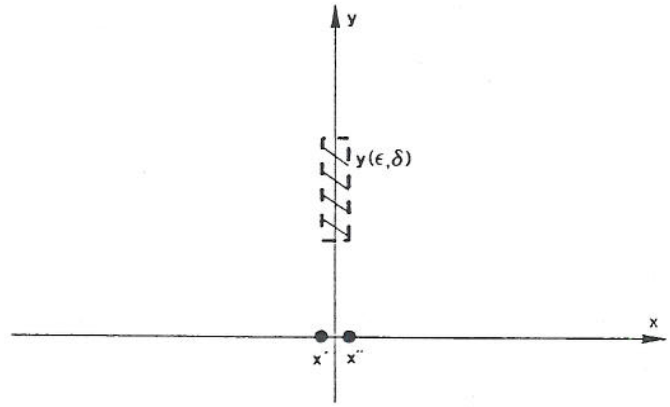

FIG. 3.3. The region $y(\varepsilon, \delta)$ and the points x' and x''.

such that $f_k(S) = f_k(S')$ *for* $1 \leqslant k \leqslant m-1$ *and any line through two points of* S' *separates* x' *from* x''.

Proof. First, rotate S if necessary to eliminate horizontal connecting lines of two points in S.

Then an affine transformation is applied to the y-values of the points of S such that the orthogonal projection onto the y-axis of the resulting points fall into the open interval $(\delta, 2\delta)$. The set S' is obtained from the latter point-set by application of a linear transformation to the x values of the points such that the required conditions are satisfied. The existence of such transformations is readily seen which completes the argument.

With Lemma 3.3 we are able to refine $S_2(n)$ considerably. To this end let $n = 2 \cdot 3^m$, for some positive integer m. We define $S_3(6) = S_1(6)$. Recall that $S_3(6)$ is in general position and realizes 12 3-sets, see Calculation 3.1. For $m > 1$, $S_3(n)$ consists of three disjoint subsets S_3^r, S_3^s, and S_3^t of $n/3$ points, respectively. S_3^r is obtained from $S_3(n/3)$ using Lemma 3.3 such that all points lie in r' and a line through any two points of S_3^r separates s' from t'. Analogously obtained are S_3^s and S_3^t from $S_3(n/3)$.

Calculation 3.4. Let $n = 2 \cdot 3^m$, m a positive integer, and let $S_3(n)$ be as described above. Then

$$f_{n/2}(S_3(n)) = n(\log_3 (n/2) + 1).$$

Proof. Due to the recursive definition of $S_3(n)$ and due to Calculation 3.2 we have $f_{n/2}(S_3(n)) = n + 3f_{n/6}(S_3(n/3))$. Straightforward calculations show that $f_{n/2}(S_3(2 \cdot 3^m)) = 2 \cdot 3^m(m+1)$ and thus $f_{n/2}(S_3(n)) = n(\log_3(n/2) + 1)$. This completes the argument.

Note that Calculation 3.4 holds only for very special positive integers n.

In what follows, the result will be generalized to arbitrary n. Then aribtrary positive integers k, with $k < n$ are considered.

THEOREM 3.5. *$f_{\lfloor n/2 \rfloor}(n) \geqslant 3^m(m+1)$, where $m = \lfloor \log_3(n/2) \rfloor$ for positive integers n.*

Proof. Let $n_1 := \lfloor n/2 \rfloor - 3^m$ and $n_2 := \lceil n/2 \rceil - 3^m$. The set $S_4(n)$ of n points is defined as the disjoint union of three sets which we call S_4^1, S_4^c, and S_4^2. S_4^c realizes the configuration of $S_3(2 \cdot 3^m)$ and we exploit Lemma 3.3 to force these $2 \cdot 3^m$ points into a small vertical strip such that a line through two points of S_4^c is nearly vertical, see Fig. 3.4. This choice of S_4^c guarantees that for each complementary pair of 3^m-sets of S_4^c there exists a line which separates those two 3^m-sets and also separates the points $(-\varepsilon, 0)$ and $(\varepsilon, 0)$, see Lemma 3.3. S_4^1 consists of n_1 points which lie strictly to the left of the lines through any two points of S_4^c. Similarly, S_4^2 contains n_2 points strictly to the right of those lines, see Fig. 3.4.

Let us now calculate the number of $\lfloor n/2 \rfloor$-sets realized by $S_4(n)$. Note that $\lfloor n/2 \rfloor = 3^m + n_1$. Due to Calculation 3.4 there are $2 \cdot 3^m(m+1)$ 3^m-sets of S_4^m and at least half of them lie to the left of a separating line. The n_1 points of S_4^1 which lie also to the left of those lines complete these $3^m(m+1)$ 3^m-sets of S_4^c to $\lfloor n/2 \rfloor$-sets of $S_4(n)$. This completes the argument.

The remainder of this section is devoted to the generalization of Theorem 3.5 to arbitrary positive integer k.

THEOREM 3.6. *$f_k(n) = f_{n-k}(n) = \Omega(n \log k)$ for positive integers n and k with $k \leqslant \lfloor n/2 \rfloor$.*

Proof. Let C denote the unit circle. Most of the points of $S_5(n, k)$ which will be described are chosen near C and some are chosen near the origin, see

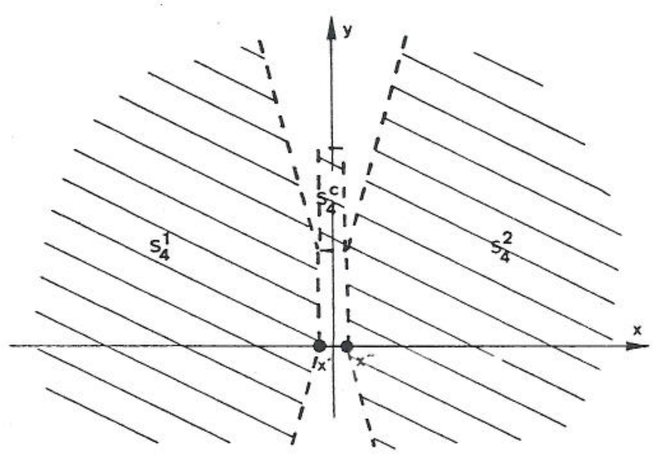

FIG. 3.4. The configuration $S_4(n)$.

Fig. 3.5. Let $j := \lfloor n/2k \rfloor$ and let $n_R := n - 2kj$. $S_5(n, k)$ is the disjoint union of point-sets $S_5^0, S_5^1, \ldots, S_5^{j-1}$, and S_5^c such that $|S_5^i| = 2k$ for $0 \leqslant i \leqslant j-1$ and $|S_5^c| = n_R$. S_5^i for $0 \leqslant i \leqslant j-1$ is chosen such that $f_k(S_5^i) \geqslant 3^m(m+1)$, $m = \lfloor \log_3 k \rfloor$ (see Theorem 3.5) and Lemma 3.3 is used to force the points into a small vertical strip. Let A^i denote the strip which contains S_5^i for $0 \leqslant i \leqslant j-1$. We translate and rotate A^i (and with A^i also S_5^i) such that A^i is contained in the interior of C and such that A^i touches C at two points and the line L^i is the symmetry axis of A^i which cuts A^i at its long sides, see Fig. 3.5. The line L^i goes through the origin and the angle from the positive x axis to L^i is $2\pi i/j$ for $0 \leqslant i \leqslant j-1$. The points of S_5^c are chosen near the origin.

Due to Lemma 3.3 the sets S_5^i for $0 \leqslant i \leqslant j-1$ can be chosen such that each line which goes through two points of S_5^i has all areas A^h with h different from i and S_5^m on one side. This guarantees that at least half of the k-sets of A^i with $0 \leqslant i \leqslant j-1$ are also k-sets of $S_5(n, k)$. By the choice of S_5^i there are at least

$$3^m(m+1)/2 \geqslant (k/3)(\log_3 k)/2$$

k-sets of A^i which survive. Hence, $f_k(S_5(n, k)) \geqslant jk(\log_3 k)/6$. The assertion follows from $jk = \Theta(n)$ which completes the argument.

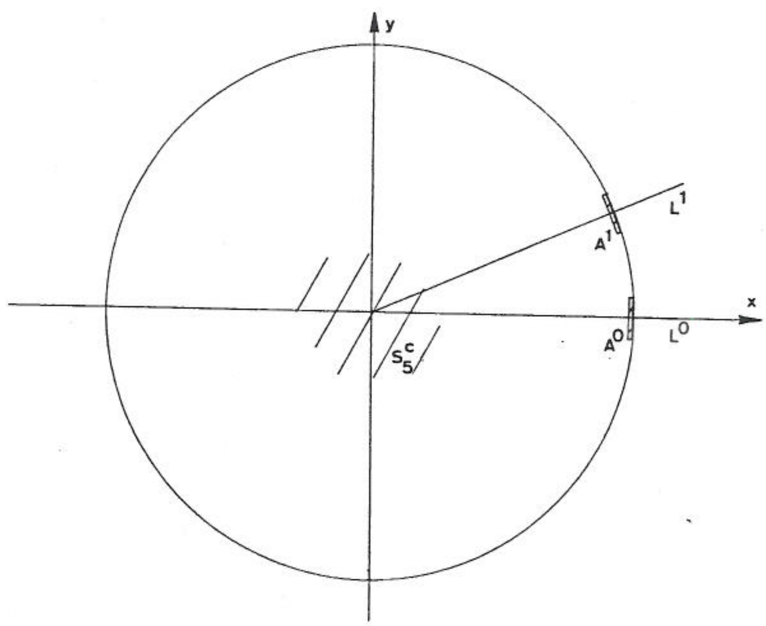

FIG. 3.5. The configuration $S_5(n, k)$.

4. An Upper Bound on the Number of k-Sets

This section presents a nontrivial upper bound on $f_k(n)$. To this end, the geometric problem is transformed into a combinatorial one. W.l.o.g. let S denote a set of n planar points in general position, see Lemma 2.4. For convenience, we label the points with the numbers $1, 2, \ldots, n$. Let L denote a directed line which is not perpendicular to any line through two points of S. Then the orthogonal projection onto L of S determines a permutation of $1, 2, \ldots, n$. As L rotates counterclockwise, say, about some fixed point it defines an infinite sequence of permutations in an obvious way. Using the terminology of Goodman and Pollack [6] we term this sequence the *circular sequence of S* and say that S *induces* its circular sequence. Evidently, the circular sequence of any set of n points has period $n(n-1)$. Two properties of circular sequences play important roles in the subsequent discussion.

Property 4.1. Two successive permutations of a circular sequence differ only by having the order of two adjacent numbers switched.

Property 4.1 follows from the assumption that the points are in general position. Thus, a permutation is obtained from its predecessor as L rotates through the direction perpendicular to the line through two points i and j. Clearly, i and j are adjacent in both permutations while their order differs. The positions of the other points cannot change simultaneously.

Property 4.2. Each one of the $n(n-1)/2$ switches occurs exactly once in any subsequence of $n(n-1)/2+1$ permutations of a circular sequence for n points.

This property corresponds to the fact that as L rotates through an angle of π it defines $n(n-1)/2+1$ permutations and rotates perpendicular through each line defined by two points of the set.

For the time being, let us ignore the close relationship between sets of points and circular sequences.

Definition 4.1. An infinite sequence of permutations of the numbers $1, 2, \ldots, n$ which satisfies Properties 4.1 and 4.2 is called an *allowable circular sequence*.

Note that the two properties imply a period of length $n(n-1)$. As an immediate consequence of Property 4.2, the ordered switch ij occurs exactly $n(n-1)/2$ steps after the ordered switch ji.

If an allowable circular sequence is induced by a point-set then the sequence is said to be *realizable*. In spite of the close relationship between sets of points and allowable circular sequences, Goodman and Pollack [6] showed that there exist allowable circular sequences which are not realizable. The distinction between point-sets and allowable circular sequences relates to the classical distinction between line-arrangements and pseudoline-arrangements, see, e.g., Gruenbaum [7].

Let us now return to our original problem, that is, to the investigation of $f_k(n)$.

DEFINITION 4.2. We call a subset A of $\{1, 2,..., n\}$ an *allowable k-set* for $1 \leqslant k \leqslant n-1$ of an allowable circular sequence C of permutations of $1, 2,..., n$ if A contains k numbers and there exists a permutation in C such that the k numbers of A occur at the leftmost k positions.

We denote by $g_k(C)$ the number of allowable k-sets realized by the circular sequence C. The maximum of $g_k(C)$, for all circular sequences C of $1, 2,..., n$, is denoted by $g_k(n)$. Evidently, $g_k(n) = g_{n-k}(n)$ and $f_k(n) \leqslant g_k(n)$. Thus, an upper bound on $g_k(n)$ is also an upper bound on $f_k(n)$.

Observation 4.1. Let C denote an allowable circular sequence of $1, 2,..., n$. Then $g_k(C)$ equals the number of switches at positions k and $k+1$ (among $n(n-1)+1$ successive permutations of C).

Clearly, each switch at positions k and $k+1$ defines a resulting allowable k-set. One also readily sees that no allowable k-set is defined more than once. Thus, we eventually consider the maximum number of switches that can place at positions k and $k+1$ during $n(n-1)+1$ successive permutations.

Let $C^k_{\max}$ denote an allowable circular sequence of $1, 2,..., n$, such that during $n(n-1)+1$ successive permutations there are $g_k(n)$ switches at positions k and $k+1$, for $k \leqslant \lfloor n/2 \rfloor$. Let C denote a subsequence of $C^k_{\max}$ which consists of $n(n-1)/2+1$ successive permutations such that at least $g_k(n)/2$ switches occur at the positions k and $k+1$. Recall that each possible (not ordered switch) occurs exactly once in C. Let $P_0 = 1, 2,..., n$ be the first permutation of C and let $Y = i, i+1,..., j$ for $1 \leqslant i \leqslant j \leqslant n$ denote a subsequence of P_0. The subsequences to the left and to the right of Y are denoted by X and Z, respectively. Hence, $P_0 = XYZ$. Let y denote the length of Y, that is, y is the number of numbers in Y. We are going to analyze the contribution of the numbers of Y to the switches at positions k and $k+1$.

LEMMA 4.2. *Let C be as defined above. Then at most $\binom{y}{2} + \min\{n-y, 2k\}$ of the switches in C at positions k and $k+1$ involve a number of Y.*

Proof. It is trivial to realize that at most $\binom{y}{2}$ of the relevant switches involve two numbers of Y. After those switches Y is totally reversed. It remains to derive a bound on those relevant switches which involve exactly one number of Y. To this end we perform certain transformations on C such that the contribution of the numbers in Y does not change and such that the switches involving two numbers of X or two numbers of Z are performed as late as possible. More specifically, we show:

Fact 1. C can be transformed such that (1) the contribution of the numbers in Y does not change and (2) no switch involving two numbers of either of X and Z is performed before the last switch (at positions k and $k+1$) involving a number of Y is completed.

Proof of Fact 1. Let f denote the smallest positive integer such that the switch leading from the fth permutation P_f in C to the $(f+1)$-th permutation P_{f+1} involves two numbers i and $i+1$ of X, say. We delete P_{f+1} from C and concatenate a permutation P^*_{f+1} at the end of C. In all permutations between P_f and P^*_{f+1}, the numbers i and $i+1$ are replaced by each other. P^*_{f+1} is chosen to differ from its predecessor only by having the order of i and $i+1$ changed. Applying this local transformation repeatedly proves the assertion of Fact 1.

Thus, in the transformed sequence of permutations, no number of X is able to move to the left before Y has completed its last contribution to the switches at positions k and $k+1$. Analogously, no number of Z is able to move to the right before this event. It follows immediately that each one of the numbers in X and Z can only once be the partner of a number in Y when it is involved in a switch at positions k and $k+1$. In addition, at most k numbers of X can move from position k to $k+1$ and at most k numbers of Z can move from position $k+1$ to k, before the last contribution of Y to the number of switches at these positions took place. This completes the argument.

THEOREM 4.3. *$f_k(n)=f_{n-k}(n)=O(nk^{1/2})$ for positive integers n and k with $k \leqslant \lfloor n/2 \rfloor$.*

Proof. We show the assertion by proving $g_k(n)=O(nk^{1/2})$. Let $P_0 = 1, 2,..., n$ be the first permutation of C as defined above. We partition P_0 into subsequences with $\lfloor k^{1/2} \rfloor$ or $\lfloor k^{1/2} \rfloor - 1$ numbers each. Due to Lemma 4.2, the contributions to the number of switches at positions k and $k+1$ of any one of those subsequences is at most $\binom{k^{1/2}}{2} + 2k < 3k$. There are at most $\lfloor n/(\lfloor k^{1/2} \rfloor - 1) \rfloor$ subsequences for $k \geqslant 4$, and at most n subsequences for $k<4$. The assertion follows since $g_k(n)$ is at most twice the quantity of switches in C at the relevant positions. This completes the argument.

5. Extensions and Discussion

First, the results and contributions of this paper are reviewed: The primary concern has been an analysis of $f_k(n)$, that is, of the maximal number of k-sets which can be realized by a set of n points in the plane. In Section 3, a lower bound is demonstrated which tells us that $f_k(n) = f_{n-k}(n) = \Omega(n \log k)$. In Section 4, an upper bound is established stating $f_k(n) = f_{n-k}(n) = O(nk^{1/2})$. Both bound improve trivial bounds which were the only bounds known before. Nevertheless, the result is not satisfying since it leaves us with quite a gap in between. We feel that the lower bound is much closer to the actual value of $f_k(n)$ and venture:

Conjecture 5.1. $f_k(n) = 0(n \log k)$ for positive integers n and k with $k \leqslant \lfloor n/2 \rfloor$.

We finally give a number of open problems raised by the discussions of this paper. The most urgent one is (1) the derivation of tighter bounds for $f_k(n)$, respectively the proof of Conjecture 5.1. Moreover, it is interesting to know how much we have lost by the transformation of the point-set problem into a combinatorial one. More specifically, (2) Do there exist positive integers n and k such that $f_k(n) < g_k(n)$, or do $f_k(n)$ and $g_k(n)$ differ even asymptotically? (3) What can be said about the sum of $f_i(S)$, for a set S of n planar points and all i in some index-set subset of $\{1, 2,..., \lfloor n/2 \rfloor\}$? These sums have applications in Edelsbrunner and Welzl [3]. At last, (4) How many k-sets can be realized by a point-set in three dimensions? Such a k-set is defined by a plane which separates it from the rest.

Note added in proof. A little while before the publication, the authors have been made aware of an article by Erdös, Lovasz, Simmons, and Straus [4], where asymptotically the same bounds for $f_k(n)$ have been established. It is interesting that the different methods in [4] lead to the same bounds.

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