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On the Maximal Number of Edges of Many Faces in an Arrangement

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Let A be an arrangement of n lines in the plane. Suppose $F_1,..., F_k$ are faces in the dissection induced by A and that F_i is a $t(F_i)$-gon. We give asymptotic bounds on the maximal sum $\sum_{i=1}^{k} t(F_i)$ which can be realized by k different faces in an arrangement of n lines. The results improve known bounds for k of higher order than $n^{1/2}$. © 1986 Academic Press, Inc.

1. INTRODUCTION AND PRELIMINARIES

An *arrangement* A of n lines is a finite set of n lines in the projective plane. The only general assumption we make is that not all lines are concurrent (which implies $n \geqslant 3$). A *face* F of such an arrangement is a connected component of its complement in the plane. Faces, respectively sets of faces, often turn out to be, "regions of equal answer" in applications in computational geometry (see Edelsbrunner and Welzl [EW] and Edelsbrunner et al. [EMPRWW]). As a consequence, these algorithms have to store all or some particular faces of an arrangement. We say a line of A *contributes* an edge to a face F if it has a nondegenerate interval in common with the closure of F. For a face F, we denote by $t(F)$ the number of lines contributing edges to its boundary, i.e., F is a $t(F)$-gon. A set $\mathscr{F}$ of faces requires memory space proportional to $\sum_{F\in\mathscr{F}} t(F)$ when it is stored in a computer.

We investigate asymptotic bounds for the maximal possible value of the sum $\sum_{F\in\mathscr{F}} t(F)$ for sets $\mathscr{F}$ of k faces in an arrangement of n lines.

Let us briefly summarize earlier results. Levi [L] observed that $t(F) \leqslant n$ for every arrangement of n lines. This was extended by Gunderson cf. Carver [Cr] who proved that $t(F_1) + t(F_2) \leqslant n + 4$ for two different faces in

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an arrangement of n lines. Finally Canham [Cn] generalized this as follows:

PROPOSITION 1.1. *If $F_1,..., F_k$ are k different faces of an arrangement of n lines, then*

$$\sum_{j=1}^{k} t(F_j) \leqslant n + 2k(k-1).$$

Furthermore, for $n \geqslant 2k(k-1)$ there exist arrangements of lines such that the upper bound is tight.

In what follows, for a natural number k and an arrangement A, we denote by $a_k(A)$ the maximal sum $\sum_{j=1}^{k} t(F_j)$ which can be realized by k different faces $F_1,..., F_k$ in this arrangement A. We denote by $a_k(n)$ the maximum of $a_k(A)$ for all arrangements A of n lines in the plane. We assume implicitly that $1 \leqslant k \leqslant n(n-1)/2+1$ since in an arrangement of n lines there are at most $n(n-1)/2+1$ different faces. In these terms Proposition 1.1 can be written as: $a_k(n) = n + 2k(k-1)$ for $n \geqslant 2k(k-1)$. For $n < 2k(k-1)$ the bound rapidly becomes rather bad, e.g., for $k = n$, we have $a_n(n) \leqslant n + 2n(n-1)$. That is, the inequality gives an estimate already worse than the trivial bound which is twice the maximal number of edges in an arrangement of n lines, namely, $2n(n-1)$.

On the other hand, determining the exact values for large k seems to be very difficult. This can be seen from the following theorem. (An arrangement of lines is *simple* if no three lines have a point in common.)

THEOREM 1.2. *Let $\psi_n = n(n-1)/2+1$, i.e., ψ_n is the maximal number of faces in an arrangement of n lines. Moreover, let $p_3(n)$ be the maximal number of triangle faces in any simple arrangement of n lines. Then*

$$a_k(n) = n(n-1)/2 + 3k - 3 \qquad \textit{for} \quad k \geqslant \psi_n - p_3(n)$$

and

$$a_k(n) < n(n-1)/2 + 3k - 3 \qquad \textit{for} \quad k < \psi_n - p_3(n).$$

Proof. First note that for any k and n there exist simple arrangements which realize $a_k(n)$. (The proof is left to the reader.) For a simple arrangement we know that the number of faces equals ψ_n (see Theorem 2.10 in Grünbaum [G2]) and the number of edges equals $n(n-1)$. Hence, if we find an arrangement of n lines such that for k faces $\mathscr{F} = \{F_1,..., F_k\}$ $\sum_{j=1}^{k} t(F_j) = a_k(n)$, then, simultaneously, the sum $\sum_{F \notin \mathscr{F}} t(F)$ realizes the minimum for $\psi_n - k$ faces. Since $t(F) \geqslant 3$ for every face F, this observation immediately implies the theorem. ∎

This theorem reveals the close relation between the functions $p_3(n)$ and $a_k(n)$: exact knowledge of $p_3(n)$ is necessary for an exact determination of $a_k(n)$. The exact value of $p_3(n)$ is known only for some small numbers n and the authors are not aware of any better bounds than $p_3 \leqslant n(n-1)/3$, for even n, and $p_3(n) \leqslant n(n-2)/3$, for odd n (consult Grünbaum [G2] for a survey of results on $p_3(n)$). This justifies the investigation of the asymptotic behaviour of $a_k(n)$ which provides a rough idea of the function.

For the purpose of analyzing the asymptotic behaviour of $a_k(n)$ we need the following notation: let $f(n)$ and $g(n)$ denote two positive valued functions on positive intergers n. We say that $f(n) = O(g(n))$ if there exists a positive constant real c such that $f(n) \leqslant cg(n)$ for all n. In addition, we say $f(n) = \Omega(g(n))$ if $g(n) = O(f(n))$, and $f(n) = \Theta(g(n))$ if $f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$. In this notation Proposition 1.1 claims that $a_k(n) = O(n + k^2)$ and that $a_k(n) = \Theta(n)$ for $k = O(n^{1/2})$, if we consider k as a function in n.

In the next section we improve this result to $a_k(n) = O(kn^{1/2} + n)$, and $a_k(n) = O(nk^{1/2})$. Moreover, we show $a_k(n) = \Omega(n^{2/3}k^{2/3} + n)$ as a lower bound. Section 3 gives a short discussion of the results.

2. Asymptotic Bounds

An easy decomposition argument together with Proposition 1.1 yields our first bound.

Theorem 2.1. $a_k(n) = O(kn^{1/2} + n)$.

Proof. For $k \leqslant (n/2)^{1/2}$ the result follows directly from Proposition 1.1. Hence, we assume that $k > (n/2)^{1/2}$. Let A be an arrangement of n lines which realizes $a_k(n)$, i.e., there are k faces $\mathscr{F} = \{F_1, ..., F_k\}$ such that $\sum_{j=1}^{k} t(F_j) = a_k(n)$. We decompose $\mathscr{F}$ into $\lceil k(2/n)^{1/2} \rceil$ disjoint subsets, each of cardinality less than or equal to $(n/2)^{1/2}$. Again due to Proposition 1.1 for each of these subsets $\mathscr{D}$ of $\mathscr{F}$ we have $\sum_{F \in \mathscr{D}} t(F) = O(n)$. As an immediate consequence

$$\sum_{F \in \mathscr{F}} t(F) = O(\lceil k(2/n)^{1/2} \rceil n) = O(kn^{1/2}),$$

which proves the theorem. ∎

This bound improves the one of Proposition 1.1 for k of higher order than $n^{1/2}$, e.g., $a_n(n) = O(n^{3/2})$ instead of $a_n(n) = O(n^2)$. Despite this fact, we still obtain the trivial upper bound of $O(n^2)$ for $k = \Omega(n^{3/2})$, k considered as a function in n. The following lemma leads to an improvement of Theorem 2.1 for $k = \Omega(n)$.

LEMMA 2.2. *Let n and m be two integers with $3 \leqslant m \leqslant n$. Then $a_k(n)\, m \leqslant a_k(m)\, n$.*

Proof. We show that $a_k(n)(n-1) \leqslant a_k(n-1)\, n$ holds, which easily implies the assertion of the lemma. For this sake, let A be a simple arrangement of n lines $\{L_1, ..., L_n\}$ which realizes $a_k(n)$. Let $\mathscr{F}$ be a set of k faces $F_1, ..., F_k$ in A such that $\sum_{j=1}^{k} t(F_j) = a_k(n)$. For L_i, $1 \leqslant i \leqslant n$, we denote by $\operatorname{contr}(L_i)$ the number of faces in $\mathscr{F}$ to which L_i contributes an edge.

Evidently, $\sum_{i=1}^{n} \operatorname{contr}(L_i) = a_k(n)$. Consequently, there is at least one line, say L_1, such that $\operatorname{contr}(L_1) \leqslant a_k(n)/n$.

Let A' denote the arrangement of lines obtained from A by removing L_1. Some of the faces in (the dissection induced by) A are still faces in A'. These faces in A are termed *invariant*. If, however, L_1 contributes an edge to a face F in A then F is merged with the other face G in A to which L_1 contributes the same edge. F is termed a *changing face*, G is termed the *gain of* F, and the face in A' which covers F and G is termed the *enlargement of* F (see Fig. 2.1).

For example, F_1 as depicted in Fig. 2.1 is an invariant face (wrt L_1) while F_2, F_3, and F_4 are changing faces. F_3 is the gain of F_4 and F_4 the one of F_3. F_3' is the enlargement of F_3.

We construct a set $\mathscr{F}'$ of k faces $F_1', ..., F_k'$ such that the loss from $\sum_{i=1}^{k} t(F_i)$ to $\sum_{i=1}^{k} t(F_i')$ is at most $\operatorname{contr}(L_1)$: (1) If F_i is an invariant face then we choose $F_i' = F_i$. (2) If F_i is a changing face such that the gain of F_i is not in $\mathscr{F}$ then F_i' is the enlargement of F_i. It is readily seen that $t(F_i') \geqslant t(F_i) - 1$. (3) If F_i and F_j are two changing faces such that one is the gain of the other then we choose for F_i' the enlargement of F_i and for F_j' any face in

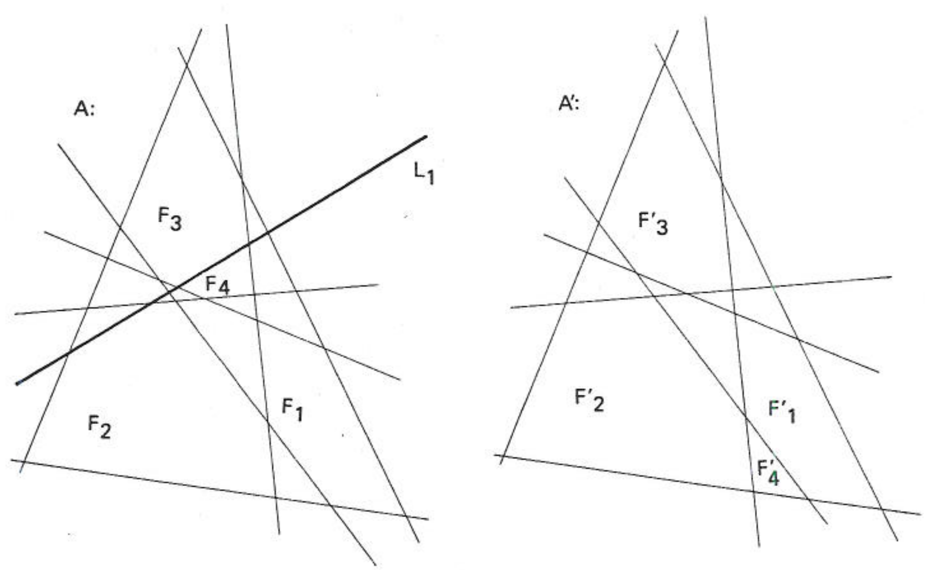

FIG. 2.1. Invariant and changing faces.

A' which has an empty intersection with each face in $\mathscr{F}$ and is not in $\mathscr{F}'$ yet. For example, F_4' in Fig. 2.1 is such a new face. Thus, $t(F_i') = t(F_i) + t(F_j) - 4$ and $t(F_j') \geqslant 3$ and therefore $t(F_i') + t(F_j') \geqslant t(F_i) + t(F_j) - 1$. This construction implies $a_k(n-1) \geqslant a_k(n) - \operatorname{contr}(L_1) \geqslant a_k(n)(1 - 1/n)$ which eventually proves the assertion stated in the first sentence of the proof. ▮

As anticipated, we exploit Lemma 2.2 for a new upper bound on $a_k(n)$.

THEOREM 2.3. $a_k(n) = O(nk^{1/2})$.

Proof. First, let $k = s(s-1)/2 + 1$, for some positive integer s. Then $a_k(s) = 2(s^2 - s)$, since $k = \psi_s$ and the number of edges in a simple arrangement of s lines equals $s^2 - s$ (recall the definition of ψ_s in Theorem 1.2). Using Lemma 2.2 we obtain

$$a_k(n) \leqslant a_k(s)\, n/s = 2n(s-1)$$

which proves the assertion for all k such that $k = s(s-1)/2 + 1$, for some positive integer s. The generalization to positive integer k for which no such s exist is trivial which completes the argument. ▮

Observe that Theorem 2.3 improves the trivial upper bound even asymptotically for all k in $O(n^{\varepsilon})$, with $0 < \varepsilon < 2$. The remainder of this section establishes a lower bound for $a_k(n)$. To this end, we need the following lemma.

LEMMA 2.4. *Let A be an arrangement of n lines. For a vertex v in the arrangement, we denote by $t(v)$ the number of lines of A passing through v. If there is a set V of vertices with $\sum_{v \in V} t(v) = a$ then there is an arrangement A' of at most $2n$ lines and a set $\mathscr{F}$ of faces in A' with $|\mathscr{F}| = |V|$ and $\sum_{F \in \mathscr{F}} t(F) = 2a$.*

Proof. Let ρ be a fixed ("small") positive real constant. For every vertex v in V we define a circle C_v with radius ρ and center v. Each line L in A which passes through at least one of the vertices in V is replaced by the lines L' and L'' parallel to L which are tangent to the circles belonging to the vertices in V which are on L.

The obtained arrangement A' has at most $2n$ lines. For $v \in V$, let F_v be the face in A' which covers the circle C_v. Clearly, we can choose ρ sufficiently small such that $t(F_v) = 2t(v)$ for all $v \in V$. Then the set of faces $\mathscr{F} = \{F_v \mid v \in V\}$ proves the assertion. ▮

Thus, the problem comes down to the analysis of arrangements of lines which realize many vertices of high degree. Such an arrangement is described in the following proof which follows lines in [F].

THEOREM 2.5. $a_k(n)=\Omega(n^{2/3}k^{2/3}+n)$.

Proof. Define $w=\lfloor k^{1/2}\rfloor$. We assume $n\geqslant 8w$ which omits only uninteresting cases. We prove the assertion by explicit construction of an arrangement A with at most $m=\lfloor n/2\rfloor$ lines.

Let $V=\{(a,b)\mid 1\leqslant a\leqslant w,\ 1\leqslant b\leqslant w\}$, a set of w^2 points with integer coordinates. For any line L, we call the number of points of V on L the *contribution of* L. To make the points of V vertices of A, let the vertical and horizontal lines that contain the columns and rows of V belong to A. Let $L(i,j,r,s)$ be the line passing through points (i,j) and $(i+r,j+s)$. To complete A, we let every line in

$$\{L(i,j,r,s)\mid 1\leqslant r\leqslant f(m,k),\ 1\leqslant i\leqslant r,\ 1\leqslant j\leqslant \lfloor w/2\rfloor,\\ \text{and } 1\leqslant s\leqslant r \text{ with } \gcd(r,s)=1\}$$

belong to A, where $f(m,k)=c_0(m/w)^{1/3}$ with c_0 a suitable positive constant to be specified later. Note that all lines in the above set are distinct. Hence we have

$$|A|=2w+\lfloor w/2\rfloor\sum_{r=1}^{f(m,k)} r\varphi(r),$$

where $\varphi(r)=|\{s\mid 1\leqslant s\leqslant r,\ \gcd(r,s)=1\}|$ denotes Euler's function. Since $\sum_{r=1}^{N} r\varphi(r)=\theta(N^3)$ (see [HW], [F]), the numer of lines in A is in $\theta(m)$, and for suitable c_0 even less or equal to m.

The contribution of $L(i,j,r,s)$ is at least $w/2r$. The overall contribution of all lines is therefore at least

$$\lfloor w/2\rfloor\sum_{r=1}^{f(m,k)}(w/2r)\,r\varphi(r)\geqslant\lfloor w/2\rfloor^2\sum_{r=1}^{f(m,r)}\varphi(r).$$

Since $\sum_{r=1}^{N}\varphi(r)=\Theta(N^2)$, (see [HW]), the overall contribution is in $\Omega(w^{4/3}n^{2/3})$ which implies the assertion by Lemma 2.4. ∎

It is interesting to note that the construction given for k vertices is asymptotically optimal as shown in [ST]. More specifically, they prove that

$$\sum_{v\in V} t(v)=O(n^{2/3}k^{2/3}+n),$$

for V a set of k vertices in an arrangement of n lines.

3. Discussion

We have investigated the asymptotic behaviour of $a_k(n)$ which designates the maximal number of edges of k faces in any arrangement of n lines in the plane. In particular, we have shown that there is a real constant c_1 such that $a_k(n) \leqslant c_1(n^{1/2}k+n)$ and $a_k(n) \leqslant c_1(nk^{1/2})$ improving the known bounds, unless k is of the order at most $n^{1/2}$. Moreover, there is a real constant c_2 such that $a_k(n) \geqslant c_2(n^{2/3}k^{2/3}+n)$. These results give a rough idea of the behaviour of $a_k(n)$. Figure 3.1 provides a graphical display of these results. This makes the gap in the current knowledge of the asymptotic behaviour of $a_k(n)$ obvious. The horizontal coordinate axis represents the logarithm of k to the base n and the vertical axis represents the logarithm of $a_k(n)$ to the base n. It should be mentioned that our upper bound arguments about arrangements of lines also go through for arrangements of pseudolines.

We have seen that the problem of determining $a_k(n)$ is closely related to determining the maximal $\sum_{v \in V} t(v)$ which can be realized by a set V of k vertices in an arrangement of n lines. Let us call the corresponding function $b_k(n)$. Lemma 2.4 showed that $b_k(n) = O(a_k(n))$; more precisely, it can be proved that $b_k(n) \leqslant a_k(n)$. We state as an open problem whether $b_k(n)$ and $a_k(n)$ have the same asymptotic behaviour, i.e., whether or not $b_k(n) = \theta(az_k(n))$. This is already known to be true for $k = O(n^{1/2})$ (see Theorem 2.18 in Grünbaum [G2]). If it would be true also for $k = \Omega(n^{1/2})$, then this would settle the problem of the asymptotic behaviour of $a_k(n)$ by the results in [ST] (see end of the previous section). Another problem in this context concerns the function $c_{k,m}(n)$ which designates the maximal sum $\sum_{F \in \mathscr{F}} t(F) + \sum_{v \in V} t(v)$ which can be realized by a set $\mathscr{F}$ of k faces and a set V of m vertices in an arrangement of n lines. Again, exact results for small k and m are known, namely,

$$c_{k,m}(n) = n + 4\binom{k}{2} + \binom{m}{2} + 2km \qquad \text{for} \quad n \geqslant 4\binom{k}{2} + \binom{m}{2} + 2km$$

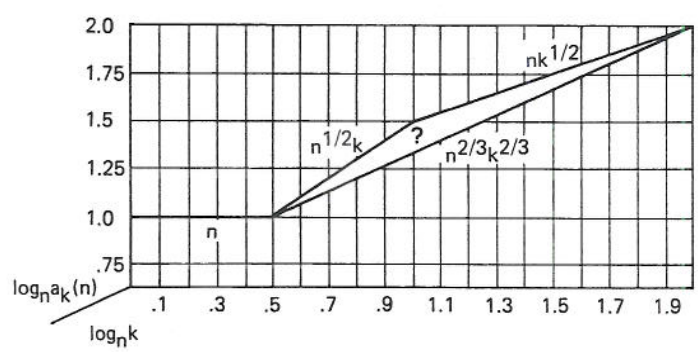

FIG. 3.1. The asymptotic behaviour of $a_k(n)$.

(see Purdy and Strommer [PS]). However, nothing seems to be known about the behaviour of $c_{k,m}(n)$ if n is smaller than required above.

Finally, a tight asymptotic lower bound for the function $p_3(n)$ (i.e., the maximal number of triangles in a simple arrangement of n lines) is of interest due to its close relation to $a_k(n)$ which is pointed out in Theorem 1.2.

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