How often can you see yourself in a convex configuration of mirrors ?

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Abstract: An edge e of a convex polygon in the Euclidean plane reflects a point p if the unique line through p which is normal to e intersects e. We prove that for every convex polygon P there is a point q reflected by at least three edges of P. We also show that for every integer $n \geq 3$ there is a convex polygon Q such that no point q is reflected by more than four edges of Q. This upper bound can be improved to three if $n \leq 6$ or if q is restricted to lie inside of Q.

1. Introduction

Let P be a convex polygon in the Euclidean plane E^2; P is called an *n-gon* if it is bounded by n edges and vertices. We assume that all edges are relatively open, that is, they do not contain their endpoints. An edge e of P is said *to reflect* a point p if the line through p, which is normal to e, meets e. Now, we define

$$\gamma(P,p)=\text{card}\{e \mid e \text{ an edge of } P \text{ which reflects } p\},$$
$$\gamma(P)=\max\{\gamma(P,q) \mid q \text{ a point in } E^2\}, \text{ and}$$
$$\gamma(n)=\min\{\gamma(P) \mid P \text{ an } n\text{-gon in } E^2\}.$$

If all edges of P are mirrors, then p "sees itself" in $\gamma(P,p)$ edges of P; here we do not consider multiple reflections and we assume that edges are transparent for reflections of other edges. We now state the main result of this paper.

Theorem 1: $\gamma(n)=3$, if $n \leq 6$, and $3 \leq \gamma(n) \leq 4$, if $n \geq 7$.

The upper bound can be improved if the points are restricted to lie inside of P.

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We therefore define

$$\delta(P)=\max\{\gamma(P,q) \mid q \text{ a point in } P\}, \text{ and}$$
$$\delta(n)=\min\{\delta(Q) \mid Q \text{ an } n\text{-gon in } E^2\}.$$

The bounds that we derive for $\delta(n)$ are tight.

Theorem 2: $\delta(n)=3$, for all integers $n \geq 3$.

The organization of this paper is as follows: Section 2 demonstrates the lower bounds of Theorems 1 and 2, and it demonstrates that the lower bound is tight for $\delta(n)$ and all values of n, and for $\gamma(n)$ if $n \leq 6$. Section 3 constructs n-gons, for every value of n, which show the upper bound of Theorem 1. Finally, Section 4 discusses the consequences of Theorem 1 to a problem which comes up in proving lower bounds on the computational complexity of some geometric problems. In addition, it briefly addresses the problem of computing $\gamma(P)$ and $\delta(P)$ if polygon P is given.

2. Lower bounds

This section gives a proof of the lower bound of Theorem 2; the lower bound of Theorem 1 then follows automatically. We also demonstrate that this lower bound is tight for $\delta(n)$ and all values of n, and for $\gamma(n)$ if $n \leq 6$. We begin with a useful characterization of when an edge reflects a point.

Let e be an edge of a convex polygon in E^2 and define

$$ref(e)=\{p \in E^2 \mid e \text{ reflects } p\}.$$

Evidently, $ref(e)$ is the union of all lines which intersect e and are normal to e; so $ref(e)$ is an open slab bounded by the two normals of e through the endpoints of e (see Fig. 3). In terms of slabs, the lower bound of Theorem 2 asserts that every convex polygon contains a point p contained in at least three slabs $ref(e)$ defined by the edges e of P. We now prove this lower bound.

Lemma 2.1: $\delta(n) \geq 3$, for every integer $n \geq 3$.

Proof: Let P be an n-gon in E^2 and let $(v_0, v_1, \ldots, v_{n-1})$ be the sequence of its vertices in counterclockwise order. We choose the indices such that v_i and v_{i+1} are the endpoints of edge e_i, for $0 \leq i \leq n-1$ and the indices taken modulo n. We will show that, for each pair of consecutive edges e_i and e_{i+1}, there is another edge e_j such that there is a point y that is reflected by all three edges. This implies the assertion. In our construction of point y, we make use of the fact that a point x in P is reflected by the edge which minimizes the distance from x. To construct y, let b_{i+1} be the angular bisector at vertex v_{i+1}, that is, b_{i+1} is the line through v_{i+1} which intersects P and bisects the

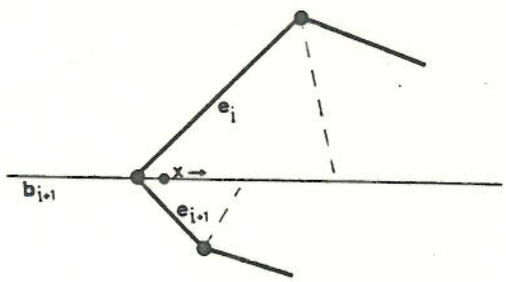


Figure 1. Construction of point y.

angle between e_i and e_{i+1} (see Fig. 1). Initially, let point x coincide with v_{i+1}. Next, we move point x continuously on b_{i+1} into P. Notice that e_i and e_{i+1} are equidistant from x as long as it belongs to $ref(e_i) \cap ref(e_{i+1})$, and that e_i and e_{i+1} are the two edges which are closest to x at the beginning or the translation of point x. Let now y be the point x at the time when it becomes equidistant to e_i, e_{i+1} and another edge e_j. This happens not later than when x hits the first of the angular bisectors at v_i and v_{i+2} (see Fig. 1). Since e_i, e_{i+1}, and e_j are also closest to point y, y is reflected by all three edges. □

By the proof of Lemma 2.1, for each pair of consecutive edges there is a third edge such that all three reflects a common point. It seems interesting to compare this to the following result.

Theorem 2.2: If every three edges of a convex polygon P reflect a common point then there is a point reflected by all edges of P.

Proof: The premise of the assertion implies that every three slabs $ref(e)$, defined for the edges e of P, have a non-empty common intersection. By virtue of Helly's theorem on convex sets in E^2, and since slabs are convex, all slabs have a non-empty intersection. □

The remainder of this section gives examples which show that the lower bound of Lemma 2.1 is best possible if, first, we restrict ourselves to points in the interior of convex polygons, and second, in the unrestricted case if $n \leq 6$.

Lemma 2.3: $\delta(n) \leq 3$, for all integers $n \geq 3$.

Proof: We give a construction of an n-gon P, with $\delta(P)=3$. Let $\epsilon>0$ be a sufficiently small real number. The vertices of P, given from left to right, are sufficiently close to the points

$$(0,-\epsilon),(1,+\epsilon),(2,-\epsilon),(3,+\epsilon),\ldots,(n-1,(-1)^n\epsilon),$$

such that P is convex (see Fig. 2). If ϵ is small enough, then a point p is reflected by three edges only if the x_1-coordinate of p is sufficiently close to an integer between 1 and n-2; otherwise, p is reflected by two edges. □

Next, we demonstrate $\gamma(n) \leq 3$ if $n \leq 6$ by showing appropriate n-gons, for $4 \leq n \leq 6$ (see Fig. 3). For each edge, we indicate the corresponding slab, and, for each sufficiently large region of the arrangement of slabs, we indicate the number of edges which reflect any point in this region. Notice the similarity between the shown quadrilateral and hexagon.

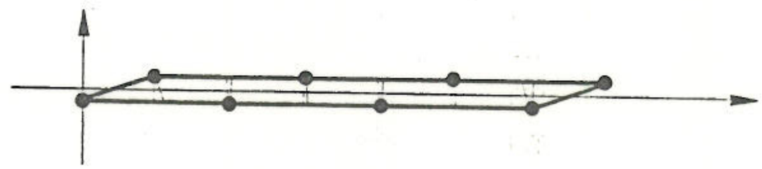
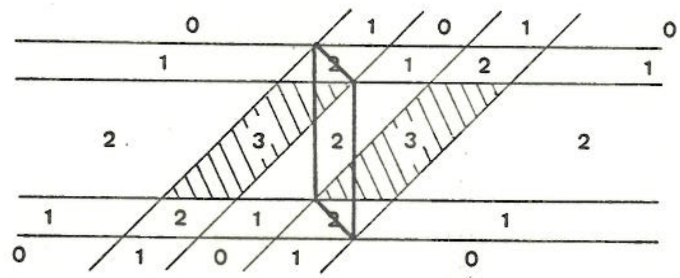
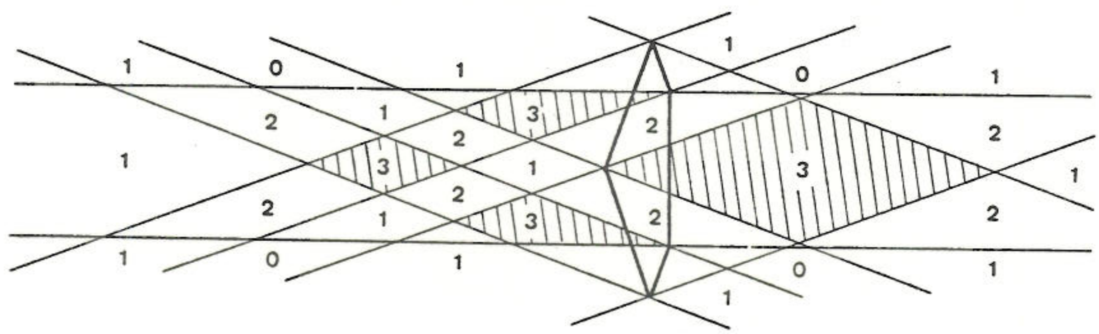


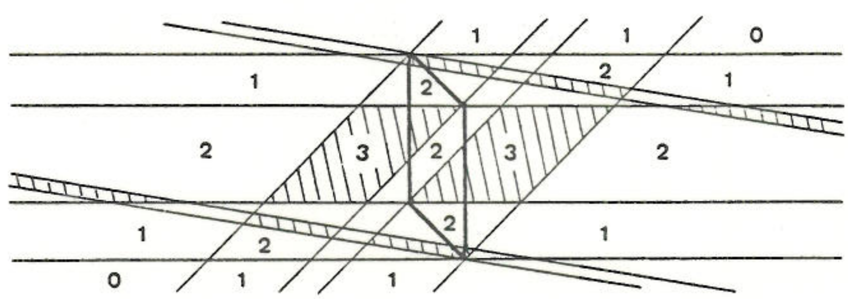
Figure 2. Construction for $n=8$.



(a) A quadrilateral P with $\gamma(P)=3$.



(b) A pentagon P with $\gamma(P)=3$.



(c) A hexagon P with $\gamma(P)=3$.

Figure 3. Polygons which show that Lemma 2.1 is tight if $n \leq 6$.

3. An Upper Bound

This section demonstrates the upper bound of Theorem 1, that is, it establishes the existence of an n-gon, for every value of $n \geq 3$, such that every point of the plane is reflected by at most four edges. To construct an n-gon, for fixed value of n, we add the edges one at a time in counterclockwise order such that no point is reflected by more than three of the first $n-1$ edges. First, we introduce some notation.

We let e_i denote the $(i+1)^{st}$ edge of construction, and we write v_i and v_{i+1} for the vertices of e_i such that v_i precedes v_{i+1} in the counterclockwise order. The positive angle between the positive x_1-axis and the translation if e_i such that v_i coincides with the origin is denoted by α_i. Furthermore, we abbreviate $ref(e_i)$ to ref_i, and we let s_i and f_i be the lines through v_i and v_{i+1}, respectively, that bound ref_i (see Fig. 4). When we add edge e_{i+1} to the chain of edges $(e_0, e_1, \ldots, e_i)$, we use the arrangement of slabs $ref_0, ref_1, \ldots, ref_i$ to guide our choice of angle and length of e_{i+1}. Angle α_{i+1} will be chosen in $[0, \pi)$, such that $\alpha_{i+1} > \alpha_i$ and such that edge e_{i+1} reflects only points reflected by at most two edges of $e_0, e_i, \ldots, e_i$, no matter how long e_{i+1}, will be; the length of e_{i+1}, will be chosen such that an appropriate angle exists for edge e_{i+2}.

Define $H_i = \{s_0, s_1, \ldots, s_i, f_0, f_1, \ldots, f_i\}$, and let (H_i) be the dissection of E^2 created by the set of lines H_i: the regions of (H_i) are the

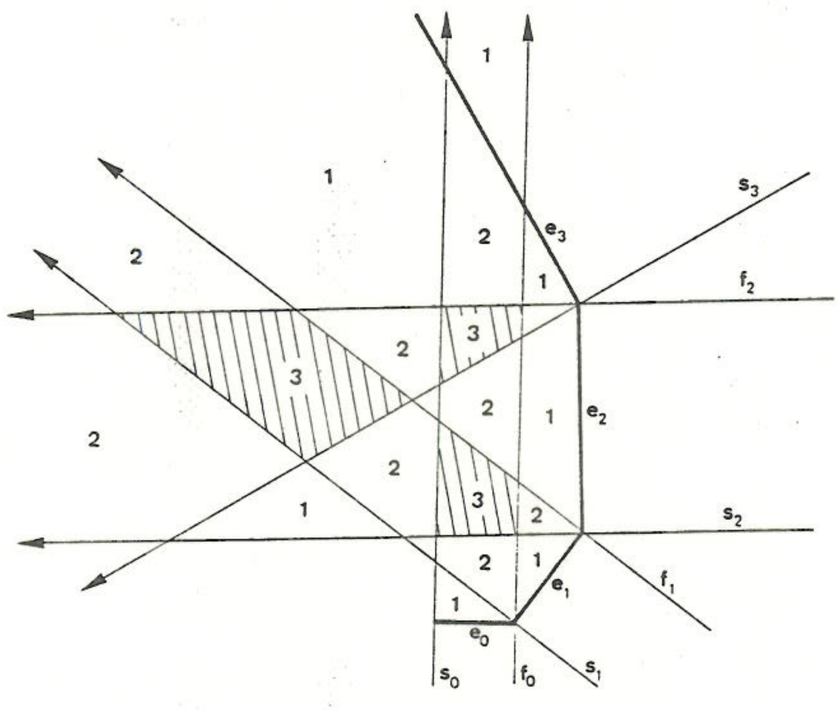


Figure 4. Construction of first four edges.

connected components of E^2 reduced by all lines in H_i. Evidently, two points p and q are reflected by the same edges, and therefore by the same number of edges, if p and q belong to the same region of (H_i). For each region r of (H_i), we let $cov(r)$ be the numbers of edges that reflect any point in r; it follows that $cov(r)$ is the number of slabs ref_j, $0 \leq j \leq i$, which cover region r. Fig. 4 indicates $cov(r)$, for each region r which is covered by at least one slab.

Lemma 3.1: A region r in (H_i) has $cov(r) \geq 2$ only if r is bounded.

Proof: If $cov(r) \geq 2$ then r is subset of the intersection of two slabs ref_j. Since $0 \leq \alpha_0 < \alpha_i < \ldots < \alpha_i < \pi$, no two slabs are parallel which implies that the intersection of any two slabs is bounded. □

We now describe the construction of an n-gon P with $\delta(P) \leq 4$. For convenience, we impose a direction on the lines s_j and f_j such that v_j is the first point of the polygon to be constructed that is hit by s_j, and v_{j+1} is the first such point hit by f_j (see Fig. 4).

Construction:

Initial step: Choose edges e_0 and e_1 such that $\alpha_0 = 0$ and $\alpha_0 < \alpha_1 < \pi$.
General step: Construct edges e_2 through e_{n-2} as follows:
for $i := 1$ **to** n-3 **do**
Step 1: Choose $\alpha_{i+1} > \alpha_i$ such that line s_{i+1} intersects the same regions of (H_{i-1}) as line f_i.
Step 2: Choose the length of edge e_{i+1} big enough such that the intersection of any two lines bounding slabs ref_0 through ref_i lies to the left of line f_{i+1}.
endfor.
Final step: Add the segment that connects vertices v_{n-1} and v_0 as edge e_{n-1} to the polygon.

Fig. 4 shows the construction up to line s_3. To prove the correctness of the construction, we show that

(i) all regions r of (H_{i-1}) which have $cov(r) \geq 2$ lie strictly to the left of line s_{i+1}, and

(ii) all regions r of (H_i) which have $cov(r) \geq 2$ lie strictly to the left of line f_{i+1}.

Assume inductively that (i) and (ii) are true for s_i and f_i. Since line f_i goes through no vertex of (H_{i-1}) (see Step 2), there exists an angle α_{i+1} such that (i) is true for line s_{i+1}: choose s_{i+1} close enough to f_i. By Lemma 3.1, all regions of (H_i) which are covered by at least two slabs are bounded. It follows that edge e_{i+1} can be chosen long enough such that (ii) is true for line f_{i+1}. This proves (i) and (ii) and shows that no region of (H_{n-2}) is covered by more than three slabs ref_j, $0 \leq j \leq n\text{-}2$. As a consequence, no region r of (H_{n-1}) has $cov(r) > 4$. This in turn implies that no point in the plane is reflected by more than four of the constructed edges. The upper bound of Theorem 1 follows.

4. Discussion and Conclusions

We proved that every convex polygon in the plane has three edges which reflect a common point, and that there are convex polygons with n edges, for every value of n, such that no point of the plane is reflected by more than four edges. In formal notation, this is equivalent to $3 \leq \gamma(n) \leq 4$. Furthermore, we showed that three is also an upper bound on the numbers of reflecting edges if $n \leq 6$ or if only points in the interior of the polygon are considered. This leaves the evaluation of the correct value of $\gamma(n)$, for $n \geq 7$, as an open problem.

Some algorithmic problems which are related to the investigations of this paper are

(i) given a convex n-gon $\mathcal{P}$, decide whether there is a point reflected by all edges of $\mathcal{P}$, and

(ii) compute the largest integer k such that there is a point reflected by k edges of $\mathcal{P}$.

As we have seen in Section 2, problem (i) can be solved by deciding whether or not the n slabs defined by the n edges of $\mathcal{P}$ have an empty common intersection. This problem can be answered in $O(n)$ time even if the slabs are not given in sorted order [PSh, Chapter 7]. To solve problem (ii), we examine all regions of the arrangement of slabs defined by the edges. The sweep algorithm of [EG] can be applied to finish this task in $O(n^2)$ time and $O(n)$ storage.

The remainder of this section relates the results of Theorems 1 and 2 to a rotation problem that came up in the investigations of lower bounds for geometric problems [B,J].

For a convex polygon $\mathcal{P}$, a point p, and an angle α, let $\rho(\mathcal{P}, p, \alpha)$ denote the number of connected components of the symmetric difference between $\mathcal{P}$ and its image under rotation around p through an angle α.

Define $\rho(P, p) = \lim_{\alpha \to 0} \rho(P, p, \alpha)$, $\rho(P) = \max\{\rho(P, q) \mid q \text{ a point in } E^2\}$, and $\rho(n) = \min\{\rho(Q) \mid Q \text{ an } n\text{-gon in } E^2\}$.

If $\alpha > 0$ is sufficiently small then an edge e intersects its image under a rotation around a point p if and only if e reflects p. Together with Theorem 1, this implies the following result.

Theorem 4.1: $\rho(n)=3$, for $n \leq 6$, and $3 \leq \rho(n) \leq 4$, for $n \geq 7$.

5. References

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