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# The Number of Extreme Pairs of Finite Point-Sets in Euclidean Spaces

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To points $p$ and $q$ of a finite set $S$ in $d$-dimensional Euclidean space $E^d$ are extreme if $\{p, q\} = S \cap h$, for some open halfspace $h$. Let $e_2^{(d)}(n)$ be the maximum number of extreme pairs realized by any $n$ points in $E^d$. We give geometric proofs of $e_2^{(2)}(n) = \lfloor 3n/2 \rfloor$, if $n \geqslant 4$, and $e_2^{(3)}(n) = 3n-6$, if $n \geqslant 6$. These results settle the question since all other cases are trivial. 

## 1. INTRODUCTION

For $S$, a set of $n$ points in $E^d$ and $h$, a half space, we call $S' = S \cap h$ a *semispace of* $S$, and a *k-set of* $S$ if $k = \text{card } S$.[1] Let $e_k(S)$ denote the number of $k$-sets realized by $S$, and define $e_k^{(d)}(n) = \max\{e_k(S) \mid S \text{ a set of } n \text{ points in } E^d\}$, for $1 \leqslant k \leqslant n-1$. The evaluation of $e_k^{(d)}(n)$ is trivial for extremely small values of $d$, $k$, or $n$: $e_k^{(1)}(n) = 2$, $e_1^{(d)}(n) = e_1(S) = n$ if $S = \text{ext } S$,[2] $e_k^{(d)}(n) = e_k^{(d-1)}(n)$ if $n \leqslant d$, and $e_k^{(d)}(d+1) = \binom{d+1}{k}$. Other trivial upper bounds follow from the one-to-one correspondence of complementary semispaces of $S$ and cells in a dual arrangement of $n$ hyperplanes in $d$-dimensional projective space. The results, e.g., in [6, 8] imply

$$\sum_{i=1}^{n-1} e_i(S) \leqslant \sum_{i=0}^{d} ((-1)^i + 1)\binom{n}{d-i},$$

[1] card $X$ denotes the cardinality of set $X$.

[2] ext $X$ contains all points of $X$ which cannot be expressed as convex combinations of other points.



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if $S$ contains $n$ points in $E^d$. Alon and Györi [1] extend this result to

$$\sum_{i=1}^{k} e_k(S) \leqslant kn,$$

if $S$ contains $n$ points in $E^2$ and $k < n/2$. Erdös, Lovasz, Simmons, and Strauss [3] proved the existence of positive constants $c_1$, $c_2$, and $n_0$ such that $e_k^{(2)}(n) \geqslant c_1 n \log_2(k+1)$ and $e_k^{(2)}(n) \leqslant c_2 n\sqrt{k}$, if $n \geqslant n_0$; the same results are derived in an independent development in [4].

This paper considers the case $k=2$ and uses *extreme pair* as a synonym for 2-set. Section 2 evaluates the value of $e_2^{(d)}(n)$ for any choice of positive integers $d$ and $n$. The geometric proof presented covers all choices of $d$: it is a new proof of the 2-dimensional result also mentioned in [4], and it is the first proof in $E^3$. Section 3 gives a lower bound for $e_3^{(2)}(n)$ and poses the investigation of $e_3^{(d)}(n)$ as an essentially open problem. It is not likely that the methods of this paper extend to $k>2$.

## 2. The Number of Extreme Pairs

This section settles the question of evaluating the maximal number of extreme pairs realized by finite point-sets in Euclidean spaces. We prove

Theorem 1. (i) $e_2^{(2)}(n) = \lfloor 3n/2 \rfloor$, *for* $n \geqslant 4$,

(ii) $e_2^{(3)}(5) = 10$ *and* $e_2^{(3)}(n) = 3n-6$, *for* $n \geqslant 6$, *and*

(iii) $e_2^{(d)}(n) = \binom{n}{2}$, *for* $4 \leqslant d \leqslant n-2$.

Note that the restriction to *simple sets of points* in $E^d$ (that is, no $d+1$ points of the set lie in a common hyperplane) is no loss of generality. We prepare the proof of Theorem 1 by two lemmas.

Lemma 2. *Let $S$ be a simple set of $n \geqslant d+3$ points in $E^d$. Any point of $S$-ext $S$ is contained in the interior of at least two simplices with the vertices chosen from $S$.*

*Proof.* Let $x$ be a point in $S$-ext $S$, if it exists. By Carathéodory's theorem (see, e.g., [2]), there are $d+1$ points $p_0, p_1, ..., p_d$ in ext $S$ with $x$ contained in int $t = \text{intconv}\{p_0, p_1, ..., p_d\}$.[3] By $n \geqslant d+3$ there is a point $y$ in $S - \{p_0, p_1, ..., p_d, x\}$. The d+1 simplices $t_i = \text{conv}\{p_0, p_1, ..., p_{i-1}, p_{i+1}, ..., p_d, y\}$, for $0 \leqslant i \leqslant d$ cover simplex $t$, and therefore $x$ belongs to int $t_i$ for at least one index $i$. ∎

[3] int $X$ denotes the interior of set $X$; conv $X$ is the set of convex combinations of all points in $X$; it is also known as the convex hull of $X$.

Note that the above argument could be used to prove the existence of $n-d-1$ simplices that contain $x$ in their interiors.

If point $x$ is in int $t$, for $t$, a simplex with vertices in $S$, then any halfspace that contains $x$ also contains at least one vertex of $t$. Therefore $x$ can form extreme pairs only with the vertices of $t$. Since two different simplices in $E^d$ share at most $d$ common vertices, Lemma 2 implies

COROLLARY 3. *Let $S$ be a simple set of $n > d+3$ points in $E^d$. Any point in $S-\text{ext}\, S$ belongs to at most $d$ extreme pairs of $S$.*

The second lemma limits the number of non-extreme points which belong to respective $d$ extreme pairs.

LEMMA 4. *Let $S$ be a simple set of points in $E^d$, $x$ a point in $S$-ext $S$ that forms extreme pairs with any one of $p_0, p_1, ..., p_{d-1}$ in $S$.*

(i) conv$\{p_0, p_1, ..., p_{d-1}\}$ *is a facet of* conv $S$.

(ii) *There is no point $y \neq x$ in $S$ with $\{y, p_0\}, \{y, p_1\}, ..., \{y, p_{d-1}\}$ all extreme pairs of $S$.*

*Proof.* We show that $x$ belongs to int $t_y = \text{intconv}\{p_0, p_1, ..., p_{d-1}, y\}$ for each point $y$ in $S - \{p_0, p_1, ..., p_{d-1}, x\}$. By Carathéodory's theorem and since $\{x, p_0\}, \{x, p_1\}, ..., \{x, p_{d-1}\}$ are all extreme pairs, there is a point $z$ in ext $S$ with $x$ in int $t_z$. For any $y$ in $S - \{p_0, p_1, ..., p_{d-1}, x, z\}$, the simplices defined by $y$ any any $d$ vertices of $t_z$ cover $t_z$. So one of these $d+1$ simplices contains $x$, and if $x$ does not belong to int $t_y$ then there is an index $i$, $0 \leq i \leq d-1$, such that $p_i$ is not a vertex of this simplex. This contradicts the extremeness of $\{x, p_i\}$.

Assertion (i) follows since all points of $S - \{p_0, p_1, ..., p_{d-1}\}$ lie on the

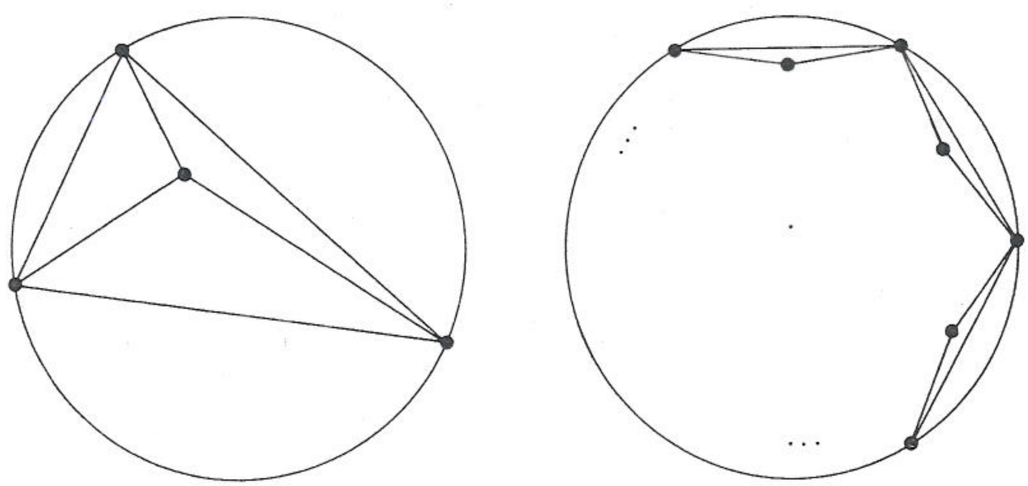

FIG. 1. Point-sets $S$ in $E^2$ with $e_2(S) = \lfloor 3n/2 \rfloor$.

same side of the hyperplane that contains $p_0, p_1, \ldots, p_{d-1}$ as $x$. Assertion (ii) follows since $x$ in int $t_y$ contradicts $y$ in int $t_x$. ∎

To prove Theorem 1, we need the following classical result in the theory of convex polytopes (see [2, 6]).

PROPOSITION 5. *Let $S$ be a simple set of $n \geqslant d+1$ points in $E^d$, with $S = \text{ext } S$. Then* conv $S$ *contains $n$, $3n-6$ edges if $d=2$, $d=3$, and at most $\binom{n}{2}$ edges if $d \geqslant 4$. The upper bound for $d \geqslant 4$ is tight.*

*Proof of Theorem* 1. Let $S$ be a simple set of $n$ points in $E^d$ with $m = \text{cardext } S$. By Proposition 5 and Corollary 3,

$$e_2(S) \leqslant m + 2(n-m) \qquad \text{if} \quad d=2,$$

$$e_2(S) \leqslant 3m - 6 + 3(n-m) \qquad \text{if} \quad d=3,$$

$$e_2(S) \leqslant \binom{m}{2} + d(n-m) \qquad \text{if} \quad d \geqslant 4,$$

provided $n \geqslant d+3$. Unless $d=2$, the upper bounds are weakest if $m=n$. If $d=2$, Lemma 4 strengthens the inequality to

$$e_2(S) \leqslant m + 2\min\{m, n-m\} + \max\{0, n-2m\}$$

which is weakest if $m = \lceil n/2 \rceil$; then $e_2(S) \leqslant \lfloor 3n/2 \rfloor$. This proves the upper bounds of Theorem 1 if $n \geqslant d+3$. The upper bounds for $n = d+2$ are trivial since there are only two essentially distinct cases: $S = \text{ext } S$ and $\text{card}(S - \text{ext } S) = 1$.

The remainder of the proof describes point-sets that prove the lower bounds of Theorem 1 in all cases. If $d=2$ and $n=4$ then one point lies in the triangle defined by the other three. If $n \geqslant 5$ then $m = \lceil n/2 \rceil$ points lie on a circle, and for each but possibly one edge there is a point sufficiently close to its midpoint but interior to the convex hull of the first $m$ points. Figure 1 illustrates both cases and indicates extreme pairs by joining segments. If $d=3$ and $n=5$ then one point is interior to the tretrahedron defined by the other four points. If $n \geqslant 6$ then $m = \text{cardext } S \geqslant (n+4)/3$ and the $n-m$ points of $S - \text{ext } S$ are chosen sufficiently close to the centroids of the $2m-4$ triangles of conv $S$, at most one point for each triangle. If $d=4$ or larger then all $n$ points can be chosen on the moment-curve $(t, t^2, t^3, t^4)$; then any pair of points is also extreme (see [2, 6] and compare with Proposition 5). ∎

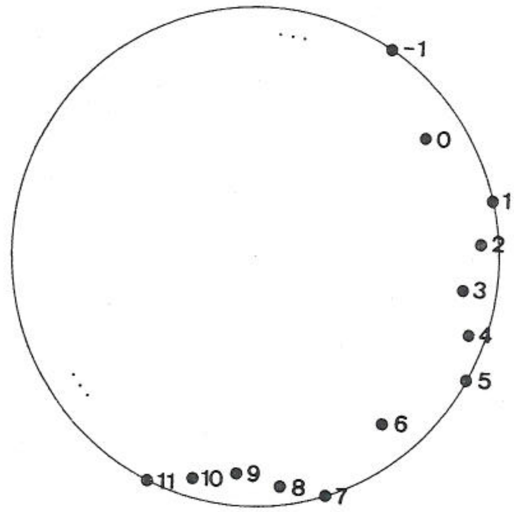


FIG. 2. Current best lower bound for $e_3^{(2)}(n)$.

## 3. Extension and Discussion

As Theorem 1 settles the evaluation of $e_k^{(d)}(n)$ for the case $k=2$, it seems natural to examine the case $k=3$. Surprisingly, there is no obvious way of extending the methods of this paper, and in fact no tight bounds are known already in $E^2$ (which might be the most difficult case, however). The current best lower bound for $e_3^{(2)}(n)$ is $\lfloor 11n/6 \rfloor$, except for a few values of $n$. For $n$ a multiple of 6, the point-sets which realize $11n/6$ 3-sets consists of groups of respective six points distributed close to a circle (as indicated in Fig. 2 $\{1, 2, \ldots, 6\}$ is a group, and the 3-sets that contain points of this group and possibly of the next group in clockwise order are: $\{1, 2, 3\}$, $\{1, 2, 4\}$, $\{1, 2, 5\}$, $\{1, 4, 5\}$, $\{2, 4, 5\}$, $\{3, 4, 5\}$, $\{4, 5, 6\}$, $\{4, 5, 7\}$, $\{5, 7, 8\}$, and $\{6, 7, 8\}$). This scheme has been discovered by Stöckl [7] running computer simulations via circular sequences, a combinatorial encoding of point-sets in $E^2$ examined in [5]. We suspect that in order to evaluate $e_3^{(2)}(n)$ a significant new insight in the general planar problem will be required.

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