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A TIGHT LOWER BOUND ON THE SIZE OF VISIBILITY GRAPHS

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The visibility graph of a finite set of line segments in the plane connects two endpoints u and v if and only if the straight line connection between u and v does not cross any line segment of the set. This article proves that 5n − 4 is a lower bound on the number of edges in the visibility graph of n nonintersecting line segments in the plane. This bound is tight.

Keywords: Computational and combinatorial geometry, visibility graph, lower bound, convex hull, Euler's relation

1. Introduction and definitions

Visibility graphs of sets of line segments in the plane have a number of applications in computational geometry. For example, the shortest path between two points in the Euclidean plane that does not cross any of a set of n line segments can be found in O(n log n + k) time after constructing the visibility graph of the line segments (see [4,6,3]), where k is the number of edges of the visibility graph. Other applications of visibility graphs can be found in [1].

Now we formally define the notion of a visibility graph of a finite set of line segments in the plane. A *line segment* ab is the convex hull of two different points a and b which are called the *endpoints* of ab. We say that two line segments s and t in the plane *cross* if their symmetric difference

$$(s \cup t) - (s \cap t)$$

consists of four connected components. Let S be a set of n nonintersecting line segments in the plane and let V be the set of 2n endpoints. Two points u and v are *visible from each other* if uv, which is a line segment not necessarily in S, does not cross any of the line segments in S. The graph $G_S = (V, E)$, with vertex set V and an edge {u, v} in E if u and v are visible from each other, is called the *visibility graph* of S.

Notice that {u, v} is an edge of G_S if u and v are endpoints of the same line segment in S. Fig. 1 shows two sets of respectively four line segments and the corresponding visibility graphs.

It is shown in [1,7] that the visibility graph of n line segments can be constructed in $O(n^2)$ time and $O(n^2)$ storage. This result has been improved in [2] to $O(n^2)$ time and O(k) storage, where k is the number of edges of the visibility graph. The time complexity of these algorithms is optimal in the worst case if all edges of the visibility graph have to be constructed explicitly. The construction of an implicit representation of a visibility graph may be possible in O(n log n) time since there are only $2^{cn \log n}$ different visibility graphs for n line segments in the plane [5]. A still unreached goal that is less ambitious than an O(n log n) time algorithm is an algorithm whose time complexity is sensitive to the output size, that is, it takes less than $\Omega(n^2)$ time if the number of edges of the

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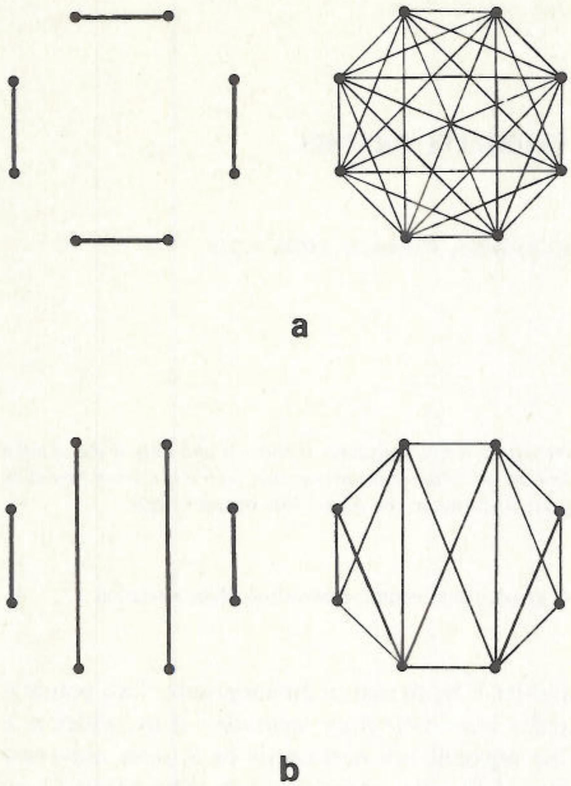

Fig. 1. Visibility graphs of two sets of line segments. (a) Dense visibility graph. (b) Sparse visibility graph.

visibility graph is subquadratic.

In this context, it is interesting to ask for the smallest and the largest number of edges of a visibility graph of n line segments. The upper bound is trivially

$$\binom{2n}{2} = 2n^2 - n$$

and it is achieved if and only if the convex hull of V is a convex polygon with 2n edges such that every other edge is a line segment in S (see Fig. 1(a)). In this article we prove that

$$5n - 4$$

is a tight lower bound on the number of edges of a visibility graph with 2n vertices. This bound is achieved if the convex hull of V is a convex polygon with 2n edges such that only two of the edges coincide with line segments in S (see Fig. 1(b)).

2. The lower bound

To simplify the forthcoming discussion, we note that we can assume without loss of generality that no three endpoints of the line segments in S are collinear; otherwise, we slightly perturb the endpoints so that no additional edges are introduced. To prove the main result, we first show two geometric lemmas.

2.1. Lemma. *If two points* u *and* v *are adjacent vertices of the convex hull of* V, *then* {u, v} *is an edge of the visibility graph.*

Proof. All points of V lie on one side of the line through u and v. As a consequence, no line segment in S can cross the line segment uv. Thus, u and v are visible from each other. □

2.2. Lemma. *If* S *contains at least two line segments, then each vertex of* G_S *is incident upon at least three edges.*

Proof. Let u be a vertex of G_S and let v be the unique other vertex such that u and v are endpoints of a common line segment s in S. By definition of G_S, {u, v} is an edge of G_S. Now, let p be a point in the relative interior of another line segment t in S such that u and p are visible from each other (see Fig. 2). Let x and y be the endpoints of t and consider the triangles Δ_x, whose vertices are u, p, and x, and Δ_y, with vertices u, p, and y. If Δ_x contains no points of V, then u and x are visible from each other. Otherwise, let w_x be the point of V in Δ_x such that the angle between line segments up and uw_x is smallest. Clearly, u and w_x are visible from each other. By the same token, {u, y} is an edge of G_S or there is a point

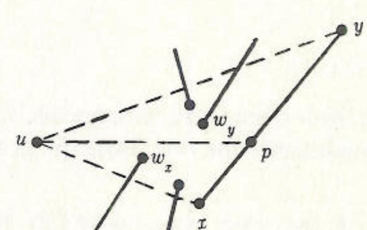

Fig. 2. Illustration of the proof of Lemma 2.2.

w_y of V in Δ_y such that $\{u, w_y\}$ is an edge of G_S. Thus, u is incident upon at least three edges of G_S. □

Now we are ready to prove the main result of this article.

2.3. Theorem. *The visibility graph of a set of* n *nonintersecting line segments in the plane contains at least* $5n-4$ *edges.*

Proof. Let S be a set of n nonintersecting line segments in the plane and let V be the set of 2n endpoints. Whenever there are two edges {u, v} and {x, y} in G_S such that uv and xy cross, then we remove one of the two edges. By repeated application of this operation, we obtain a planar graph T. The embedding of T in the plane that represents each edge {u, v} by the line segment uv is a triangulation of the convex hull of V. Let f_0, f_1, and f_2 denote the number of vertices, edges, and faces of T, respectively. Trivially, we have

$$f_0 = 2n,$$

and f_1 is at most equal to the number of edges of G_S. By Euler's relation, we also have

$$f_0 - f_1 + f_2 = 2.$$

Now, each face is bounded by exactly three edges except for the unbounded face which is bounded by $b \geqslant 3$ edges and vertices. Thus,

$$\tfrac{1}{3}(2f_1 - b) = f_2 - 1$$

and, therefore,

$$f_0 - f_1 + \tfrac{2}{3}f_1 - \tfrac{1}{3}b = 1$$

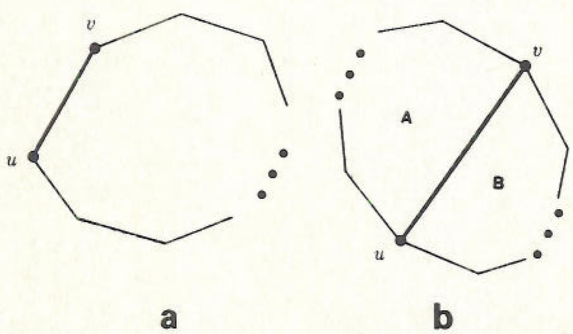

Fig. 3. The convex hull of V has at least n+1 vertices.

or, equivalently,

$$f_1 = 6n - b - 3.$$

Next, we distinguish two cases: $b \leqslant n$ and $b \geqslant n+1$. If $b \leqslant n$, then we have

$$f_1 \geqslant 5n - 4,$$

which implies that G_S has at least $5n-4$ edges. Otherwise, that is, if $b \geqslant n+1$, there is at least one line segment uv in S such that both u and v are vertices of the convex hull of V (see Fig. 3). The remainder of the argument is inductive, that is, we assume that the visibility graph for any $m < n$ nonintersecting line segments in the plane has at least $5m-4$ edges. This assumption is trivially true for $m = 1$. Assume first that uv coincides with an edge of the convex hull of V (see Fig. 3(a)). Line segment uv does not cross any line segment xy, with x and y any two points in V. It follows that no new visible pair of endpoints is created when we remove uv from S. By Lemma 2.2, u and v are incident upon at least five edges of G_S. Thus, G_S has at least five edges more than $G_{S-\{uv\}}$. Finally, $G_{S-\{uv\}}$ has at least

$$5(n-1) - 4$$

edges, by induction hypothesis, which implies that G_S has at least

$$5n - 4$$

edges. Next, we assume that uv does not coincide with an edge of a convex hull of V (see Fig. 3(b)). Let $A \subseteq S$ be the set of line segments on one side of the line through u and v and define $B = S - A - \{uv\}$. There is no edge {x, y} in G_S with x an endpoint of a line segment in A and y an endpoint of a line segment in B, since line segments xy and uv would cross otherwise. Consequently, each edge of G_S is an edge of $G_{A\cup\{uv\}}$ or of $G_{B\cup\{uv\}}$, and {u, v} is an edge of both visibility graphs. Let $n_a - 1$ and $n_b - 1$ be the cardinalities of A and B, respectively. We have $n_a + n_b = n + 1$ with $n_a < n$ and $n_b < n$ since both A and B are nonempty. By induction hypothesis, $G_{A\cup\{uv\}}$ and $G_{B\cup\{uv\}}$ together have at least

$$(5n_a - 4) + (5n_b - 4) = 5(n+1) - 8$$

edges one of which is counted twice. Consequently, G_S has at least

$$5(n + 1) - 9 = 5n - 4$$

edges. □

3. Remarks

In this article, we proved that the visibility graph of n nonintersecting line segments in the plane has at least 5n − 4 edges. It is crucial for the proof of this bound that an edge {u, v} is in the visibility graph if and only if the line segment uv does not cross any line segment of the given set. According to another natural definition, {u, v} is an edge of the visibility graph if uv is a line segment of the set or if the relative interior of the line segment uv is disjoint from any line segment in the given set. In this case, the lower bound on the number of edges is trivially 2n − 1, since it is connected. This lower bound is achieved if all n line segments lie on a common line. If we assume that no three endpoints are collinear, then both definitions of visibility graphs are equivalent.

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