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On the Number of Furthest Neighbour Pairs in a Point Set

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Introduction. Consider a configuration N of points in the Euclidean plane, labeled $1, 2, \ldots, n$. A point j is a *furthest neighbour* of i if $d(i, j) = \max_{1 \leqslant k \leqslant n}\{d(i, k)\}$, where d is the Euclidean distance function. An ordered pair of points (i, j), $i, j \in N$ is a *furthest neighbour pair* of N if j is a furthest neighbour of i. We let $M(N)$ denote the number of furthest neighbour pairs defined by N. Each point has at least one furthest neighbour, so we have $M(N) \geqslant n$. Our interest is in $M(n)$, the largest number of furthest neighbour pairs possible in a configuration of n points. Avis [1] proves $M(n) = 3n - 3$ for even $n \geqslant 4$ and $3n - 4 \leqslant M(n) \leqslant 3n - 3$ for odd $n \geqslant 5$. Counting furthest neighbour pairs can be generalized to counting repeated distances with other restrictions, as discussed in [2].

This note corrects Avis' proof for the even case and improves the result to $M(n) = 3n - 4$ for odd $n \geqslant 5$. Also, we show that a convex set of $n \geqslant 3$ points has at most $2n$ furthest neighbour pairs and that this bound is tight. Finally, we prove

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that the number of furthest neighbour pairs of n points in three dimensions is subquadratic if no three points are collinear; this is the case if the set is convex.

Results in Two Dimensions. First, we introduce some terminology which borrows from [1]. Let n_i be the number of furthest neighbours of i and let m_i be the number of points j such that (j, i) is a furthest neighbour pair. We use the distance r_i of i to its furthest neighbours as a radius to define circle C_i and disk D_i centered at i. Each disk D_i contains all points $j \in N$, so N lies in $\bigcap_{i=1}^{n} D_i$. Thus n_i can also be defined as the number of points on C_i and m_i as the number of circles passing through i.

Avis proves $M(N) \leqslant 3n - 3$ by considering the circles C_i, $i = 1, 2, \ldots, n$, in order of non-decreasing radius, that is, $r_i \leqslant r_{i+1}$. Each point that lies on C_k is one of two types, either one which lies on a C_j, $j < k$ or one which does not. We let h_k be the number of points of the first type and f_k the number of the second type. By definition,

$$M(N) = \sum_{k=1}^{n} f_k + \sum_{k=2}^{n} h_k.$$

Since each point can be intersected for the first time only once, $\Sigma_{k=1}^{n} f_k \leqslant n$. This coupled with Lemma 1 gives $M(N) \leqslant 3n - 2$, which can be improved to the stated result.

Unfortunately, the original proof of Lemma 1 given in [1] uses an incorrect argument. We present an alternate proof:

LEMMA 1. *$rh_k \leqslant 2$ for $k = 2, 3, \ldots, n$.*

Proof. Assume $h_k \geqslant 3$. A point is of the first type with respect to C_k only if it lies on the boundary B of $\bigcap_{i=1}^{k-1} D_i$. Thus C_k intersects B in at least three points. Since $r_i \leqslant r_k$ for all $i < k$, at each intersection at least one side of C_k is outside B. Consequently, the intersection points divide C_k into arcs at least two of which lie outside B. The smallest of these arcs spans an angle $\theta < \pi$ with respect to k. Thus, there is an index $i < k$ such that C_i intersects C_k in two points that belong to this arc. So an arc of C_k spanning an angle greater than π must be within C_i. This is a contradiction since $r_i \leqslant r_k$.

We use two results to prove our main theorem. The first is a technical lemma, the other a bound on $M(N)$ for convex sets N. In this context, a set of points is said to be *convex* if each point is a vertex of the convex hull of the set.

LEMMA 2. *For a set N of $2k$ points on a circle C, we have $M(N) < 4k$.*

Proof. Assume the converse, that there exists a circular arrangement N of $2k$ points where $M(N) \geqslant 4k$. Since any two different circles intersect in at most two points, each point has at most two furthest neighbours on C. Thus $M(N) = 4k$.

Let u and v be furthest neighbours of s on C and c be the center of C. When the points are angularly ordered around c, u and v must be consecutive points. If there were to be another point w between them, either $d(s, w) > d(s, v)$ or $d(s, u) \neq d(s, v)$. If $M(N) = 4k$, a point u can be furthest neighbour of another point only if this other point has another furthest neighbour v. Now, v can only be the predecessor or the successor of u. Consequently, u is furthest neighbour of exactly two points. These two points, s and t, are also the furthest neighbours of u.

Otherwise, u has a furthest neighbour r between s and t which implies that u is also furthest neighbour of r, a contradiction. As a consequence, the points must be equally spaced around C. However, the line through s and c will intersect another point if an even number of points are equally spaced, so s has only one furthest neighbour, a contradiction. Therefore, $M(N) < 4k$.

THEOREM 3. *For a convex set N of $n \geqslant 3$ points, $M(N) \leqslant 2n$ and the bound is tight.*

Proof. Label the points according to their counterclockwise order around the convex hull of N. If $v_1, v_2, \ldots, v_m$ are the successive furthest neighbours of point i, we show that only v_m can be a furthest neighbour of $i + 1$. Consider the perpendicular bisector between v_1 and v_2, which passes through i. Point $i + 1$ is on the same side of the bisector as v_1, which means $d(i + 1, v_1) < d(i + 1, v_2)$ and similarly $d(i + 1, v_j) < d(i + 1, v_{j+1})$, $j < m$.

We maintain two pointers as we move around N, i and v, where v is a furthest neighbour of i. Initially $i = 1$ and v is the first furthest neighbour of i. Each step moves v forward until it is the n_i^{th} furthest neighbour, when i is advanced. Neither i nor v can retreat. We stop when $i = n$ and v is the n_n^{th} neighbour of n, which must be less than or equal to the initial value of v. The situation is shown in Figure 1. Each step corresponds to a furthest neighbour pair, and the only pair not defined by a step is represented by the initial configuration. Since i is advanced $n - 1$ times, and v at most n times, adding the initial pair gives $M(N) \leqslant n + (n - 1) + 1 = 2n$.

To show that the upper bound is tight, take the vertices of an equilateral triangle and, for each vertex i, draw the shorter circular arc centered at i that connects the other two vertices. Pick the three vertices together with $n - 3$ arbitrary points on the three arcs to form N; then $M(N) = 2n$.

With these results we can prove the exact value of $M(n)$. For even n the result is the same as in Avis [1]. For completeness and since it is requires no extra effort we include this case.

THEOREM 4. *$M(n) = 3n - 3$ if $n \geqslant 4$ is even and $M(n) = 3n - 4$ if $n \geqslant 5$ is odd.*

Proof. We perform a case analysis on m, the number of points that are not vertices of the convex hull of N.

(1) Let $m = 0$. The n points form a convex set. By Theorem 3, $M(N) \leqslant 2n$.

(2) Let $m = 1$. Exactly one point i violates convexity. In this case, i cannot be furthest neighbour of any other point. Either $n_i = n - 1$ or $n_i < n - 1$. The first possibility implies the $n - 1$ points lie on a circle. Thus $M(N) \leqslant n - 1 + 2(n - 1)$ which implies the result for even n. If n is odd, then by Lemma 2, $M(N) < n_i + 2(n - 1) \leqslant 3n - 3$. Similarly, if $n_i < n - 1$ and the rest of the points are convex, by Theorem 3:

$$M(N) \leqslant n_i + 2(n - 1) \leqslant n - 2 + 2(n - 1) \leqslant 3n - 4.$$

(3) Let $m \geqslant 2$. A total of m points violate convexity. Since none of the m points can be the furthest neighbour of any other point,

$$M(N) = \sum_{k=1}^{n} n_k = \sum_{k=1}^{n} f_k + \sum_{k=2}^{n} h_k \leqslant (n - m) + 2(n - m) \leqslant 3n - 4.$$

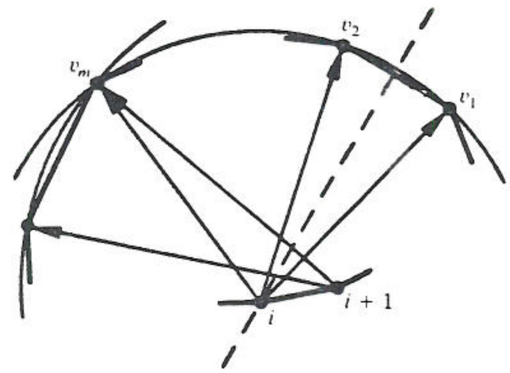

FIG. 1. Counting furthest neighbours in a convex set.

A point set N realizing this bound can be constructed. If n is even arrange $n-1$ points equally spaced along a circle, one point c in the center. The center point has $n-1$ furthest neighbours, and the $n-1$ points have two furthest neighbours each. Thus, $M(N)=3n-3$. If n is odd add a point along a radius between c and any other point to the construction for $n-1$ points. Now, the center point has $n-2$ furthest neighbours, and again every other point has two furthest neighbours; so $M(N)=3n-4$.

Extensions to Three Dimensions. In three dimensional Euclidean space, we can construct a set N of n points with $M(N) \geqslant (n^2+2n)/4$. Choose half of the points on a circle C and choose the other points on the perpendicular line l through the center of C. If n is odd, then let the number of points on C be one more than on l. The points on l can be picked such that for each i on l and for each j on $C(i, j)$ is a furthest neighbour pair. Only the lower order terms of this construction can be improved (see [**2**]).

Note that the above construction uses about $n/2$ points which are collinear. Interestingly, collinear points are necessary to obtain a quadratic number of furthest neighbour pairs.

THEOREM 5. *$M(N) \leqslant c \cdot n^{5/3}$ if N is a set of n points in three dimensions such that no three points are collinear.*

Proof. Let r_i be the distance from point i to its furthest neighbours and let S_i be the sphere with radius r_i and center i. Three such spheres intersect in at most two points since their centers are not collinear.

Define the 3-regular multi-hypergraph H with node set N that contains a hyperedge $\{i, j, k\}$ m times if points i, j, and k share m common furthest neighbours. Since $|S_i \cap S_j \cap S_k| \leqslant 2$, each hyperedge can occur at most twice which implies that H has at most $2\binom{n}{3}$ hyperedges. Recall that point i is furthest neighbour of m_i other points. Thus, i contributes $\binom{m_i}{3}$ hyperedges to H. This implies

$$\sum_{i=1}^{n} \binom{m_i}{3} \leqslant 2\binom{n}{3}.$$

By the Cauchy-Schwartz inequality (see [3]), we infer

$$\sum_{i=1}^{n} m_i \leqslant 2^{1/3} n^{5/3} + o(n^{5/3}).$$

For example, if N is a convex set then no three of its points are collinear and therefore, $M(N) = O(n^{5/3})$. It is not known whether or not $M(N)$ can be superlinear in this case.

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