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CIRCLES THROUGH TWO POINTS THAT ALWAYS ENCLOSE MANY POINTS*

ABSTRACT. This paper proves that any set of n points in the plane contains two points such that any circle through those two points encloses at least $n(1/2 - 1/\sqrt{12}) + O(1) \approx n/4{\cdot}7$ points of the set. The main ingredients used in the proof of this result are edge counting formulas for k-order Voronoi diagrams and a lower bound on the minimum number of semispaces of size at most k.

1. Introduction

Hayward [4] proves that any set of n points in the plane contains a pair of points with the property that any circle through the two points encloses at least $\lceil 5/84(n-2) \rceil$ of the points. Here we say that a circle *encloses* a point if the point belongs to the open bounded region whose boundary is the circle. We assume that the n points are in general position, that is, no three points are collinear and no four are cocircular. The problem was introduced in [7] and linear lower bounds slightly weaker than the one in [4] were subsequently proved in [5] and [1]. [5] also shows that there is a set of n points with the property that for any two points there is a circle through the two points enclosing at most $\lceil n/4 \rceil - 1$ points, and they completely solve the case for convex point sets† – in this case $[n/3] - 1$ is an upper and a lower bound.

This note improves the lower bound for general sets of n points to

$$\frac{n-3}{2} - \left(\frac{(n-2)^2 - 1}{12}\right)^{1/2} \approx \frac{n}{4{\cdot}7}.$$

That is, we prove that there are always two points such that any circle through the two points encloses at least

$$\frac{n-3}{2} - \left(\frac{(n-2)^2 - 1}{12}\right)^{1/2} \approx \frac{n}{4{\cdot}7}$$

of the points. This is proved in Section 2. The proof makes use of tight bounds of the number of edges in higher-order Voronoi diagrams as derived in [6]. It also uses a tight lower bound on the number of semispaces of size at most k of a set of n points in the plane; this bound is proved in Section 3. Incidentally, our

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† A finite point set is said to be *convex* if it is the set of vertices of a convex polygon.

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technique yields an alternate proof of the tight lower bound for convex point sets.

2. Proof of the lower bound

Our approach to the problem is quite different from the one in [4]. For any two points $p, q \in S$ we draw the perpendicular bisector and consider the pencil of circles through p and q (see Figure 1). For $n-2$ such centers the circle goes through a third point of S and we mark these centers on the line. The line is now cut into $n-1$ open intervals two of which are unbounded. For each such interval, ι, there is a unique non-negative integer, $\rho(\iota)$, such that any circle through p and q whose center lies on ι encloses $\rho(\iota)$ points of S. As we move the center from ι to an adjacent interval, κ, we get $\rho(\kappa) = \rho(\iota) - 1$ or $\rho(\kappa) = \rho(\iota) + 1$ depending on whether the circle corresponding to the moving center passes a point that was inside or outside the circle. This is all illustrated in Figure 1. The problem is thus to prove that there is always a perpendicular bisector whose smallest ρ value is at least

$$\frac{n-3}{2} - \left(\frac{(n-2)^2 - 1}{12}\right)^{1/2}.$$

The intervals on the perpendicular bisectors are actually edges of the so-called higher-order Voronoi diagrams of S. For any integer k, $1 \leqslant k \leqslant n-1$, the *kth-order Voronoi diagram* of S, denoted as $\mathscr{V}_k(S)$, is the decomposition of the plane into maximal regions such that the k nearest neighbors of any two

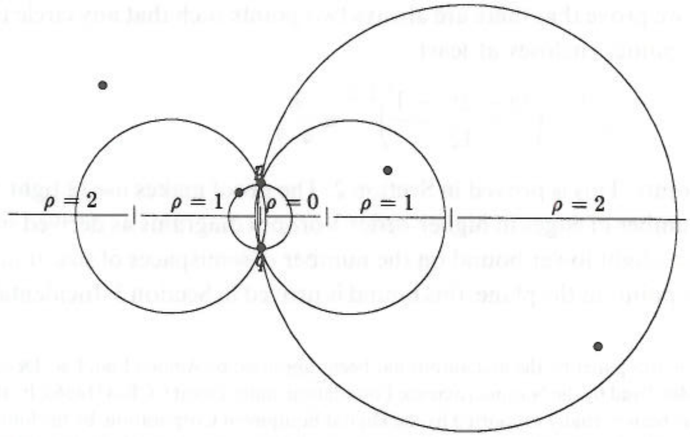

Fig. 1.

points x and y in the same region are the same. Here x and y are any two points in a region not necessarily in S, and the k *nearest neighbors* of x, say, are the k points of S whose distances from x are the k smallest. As observed in [8] (see also [2], [6]) an interval ι of a perpendicualr bisector is an edge of $\mathscr{V}_k(S)$ if and only if $\rho(\iota) = k - 1$. Our problem is thus to prove that there is always a perpendicular bisector of two points that contains no edge of $\mathscr{V}_k(S)$, for

$$k < \frac{n-3}{2} - \left(\frac{(n-2)^2 - 1}{12}\right)^{1/2} + 1.$$

To prove this, we make use of the following counting result proved in [6] (see also [2]). Here an *i-set* of S is a subset of i points that can be defined as the intersection of S with a half-plane; the number of i-sets of S is denoted by $e_i(S)$.

PROPOSITION 1. *Let S be a set of n points in the plane where no three points are collinear and no four points are cocircular. The number of edges of the kth-order Voronoi diagram of S is*

$$(6k-3)n - 3k^2 - e_k(S) - 3 \sum_{i=1}^{k-1} e_i(S),$$

for $1 \leqslant k \leqslant n - 1$.

It is amazing that the formula in Proposition 1 gives the exact number of edges of $\mathscr{V}_k(S)$ only as a function of n and k and the number of i-sets of S, for $1 \leqslant i \leqslant k$. To maximize the number of edges we have to minimize the number of i-sets, $1 \leqslant i \leqslant k$. Avoiding the interruption of the flow of the current argument, we postpone the proof of a tight lower bound on this minimum until Section 3. There, the result will be stated as Theorem 3.

We now derive our result using Proposition 1. Roughly, the idea is to prove that the first k Voronoi diagrams do not have enough edges to cover all $\binom{n}{2}$ perpendicular bisectors. Here, we say that the first k Voronoi diagrams *cover* a perpendicular bisector of two points if it contains at least one edge of this bisector. First we introduce some notation. Let ε_k be the number of edges of $\mathscr{V}_k(S)$, that is,

$$\varepsilon_k = (6k-3)n - 3k^2 - e_k(S) - 3 \sum_{i=1}^{k-1} e_i(S).$$

The unbounded regions of $\mathscr{V}_k(S)$ stand in one-to-one correspondence with the k-sets of S. Since the number of unbounded regions is equal to the number of unbounded edges of $\mathscr{V}_k(S)$, we have $v_k = e_k(S)$, with v_k the number of unbounded edges of $\mathscr{V}_k(S)$. Notice that a bounded edge has two adjacent edges on the same line whereas an unbounded edge has only one such adjacent edge.

Picturally, we say that a bounded edge has 'two hands' and an unbounded edge has 'one hand'. If we put down two adjacent edges then one hand of one edge holds one hand of the other edge. The second hands of both edges (if there are any) are *free* (see Figure 2).

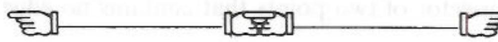

Fig. 2.

The total number of hands of all k-order edges is $2\varepsilon_k - v_k$. If we put down all edges of $\mathscr{V}_k(S)$ we thus get $\phi_1 = 2\varepsilon_1 - v_1$ free hands. All edges adjacent to 1-order edges belong to $\mathscr{V}_2(S)$. As we put down all edges of $\mathscr{V}_2(S)$ we add $2\varepsilon_2 - v_2$ hands but ϕ_1 of these hands are used to satisfy the free hands of $\mathscr{V}_1(S)$. This leaves $\phi_2 = (2\varepsilon_2 - v_2) - (2\varepsilon_1 - v_1)$ free hands after putting down all order-2 edges. Similarly, ϕ_2 hands of 3-order edges are used to satisfy those free hands which leaves $\phi_3 = (2\varepsilon_3 - v_3) - (2\varepsilon_2 - v_2) + (2\varepsilon_1 - v_1)$ free hands. In general, the number of free hands after putting down all edges of $\mathscr{V}_i(S)$, $1 \leqslant i \leqslant k$, is

$$(*) \qquad \phi_k = \sum_{i=1}^{k} (-1)^{k-i}(2\varepsilon_i - v_i).$$

It is straightforward to check that ϕ_k is indeed positive but we hope that the critical reader does not take our word for it.

In order to get an upper bound on the number of covered lines, we count the number of connected components formed by the edges that we put down on the lines, where a *connected component* is a maximal connected subset of a line that is the union of the closures of edges already put down. Clearly this is an upper bound on the number of covered lines since a line is covered only if it contains at least one connected component. Let λ_k be the number of connected components formed by all edges on $\mathscr{V}_1(S)$ through $\mathscr{V}_k(S)$. We derive a formula for λ_k by distinguishing three cases as we put down k-order edges. If an edge satisfies two hands at the same time it decreases the number of connected components by one (see Figure 3(a)). If it satisfies one hand it extends an existing connected component; it also generates a new free hand (see Figure 3(b)). If the edge is not adjacent to any $(k-1)$-order edge and thus satisfies no free hand, then it creates a new connected component and two new free hands (see Figure 3(c)).

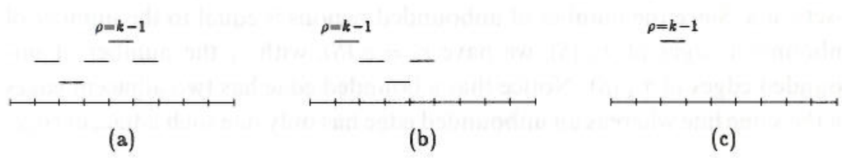

Fig. 3.

Let ε_a, ε_b, and ε_c be the number of k-order edges of the first, second, and third type. The number of connected components after putting down all edges of $\mathscr{V}_k(S)$ is

$$\lambda_k = \lambda_{k-1} + \varepsilon_c - \varepsilon_a.$$

We also have $\varepsilon_a + \varepsilon_b + \varepsilon_c = \varepsilon_k$ and $\phi_{k-1} = 2\varepsilon_a + \varepsilon_b$ which allows us to rewrite this formula as

$$(**) \qquad \lambda_k = \lambda_{k-1} + \varepsilon_k - \phi_{k-1}.$$

If we iterate $(**)$ we get

$$(***) \qquad \lambda_k = (\varepsilon_k - \phi_{k-1}) + (\varepsilon_{k-1} - \phi_{k-2}) + \cdots + (\varepsilon_2 - \phi_1) + \varepsilon_1$$

$$= \sum_{i=1}^{k} \varepsilon_i - \sum_{i=1}^{k-1} \phi_i.$$

From $(*)$ we get $\phi_i + \phi_{i-1} = 2\varepsilon_i - v_i$ which allows us to rewrite $(***)$ as

$$\lambda_k = \sum_{i=1}^{k} \varepsilon_i - [(2\varepsilon_{k-1} - v_{k-1}) + (2\varepsilon_{k-3} - v_{k-3}) + \cdots +$$

$$+ (2\varepsilon_2 - v_2)]$$

if k is odd and the same sum through $(2\varepsilon_1 - v_1)$ is k is even. After rearranging the terms we get

$$\lambda_k = \sum_{i=1}^{k} (-1)^{k-i} \varepsilon_i + (v_{k-1} + v_{k-3} + \cdots + v_{2\text{or}1}).$$

From Proposition 1 we get

$$\varepsilon_k - \varepsilon_{k-1} = 6n - 6k + 3 - e_k(S) - 2e_{k-1}(S).$$

In the final calculation we distinguish the case that k is even from k odd. If k is even we get

$$\lambda_k = \sum_{i=1}^{k/2} [6n - 12i + 3 - e_{2i}(S) - e_{2i-1}(S)]$$

$$= (6n + 3)\frac{k}{2} - 12\binom{k/2 + 1}{2} - \sum_{i=1}^{k} e_i(S)$$

$$= 3kn - \frac{3k^2}{2} - \frac{3k}{2} - \sum_{i=1}^{k} e_i(S).$$

If k is odd λ_k is

$$\lambda_k = \sum_{i=1}^{(k-1)/2} [6n - 6(2i+1) + 3 - e_{2i+1}(S) - e_{2i}(S)] + \varepsilon_1$$

$$= (6n+3)\frac{k-1}{2} - 12\binom{(k+1)/2}{2} - 6\frac{k-1}{2} - \sum_{i=2}^{k} e_i(S) +$$

$$+ 3n - 3 - e_1(S) = 3kn - \frac{3k^2}{2} - \frac{3k}{2} - \sum_{i=1}^{k} e_i(S).$$

Thus we get the same result in both cases. If we now use Theorem 3 we get

$$\lambda_k \leqslant 3kn - \frac{3}{2}(k^2+k) - \frac{3}{2}(k^2+k) = 3kn - 3k^2 - 3k.$$

To finally solve our problem we compute the largest k such that the number of connected components is at least one short of the total number of lines. Thus we would like to know the largest k such that

$$\lambda_k \leqslant 3kn - 3k^2 - 3k < \binom{n}{2}.$$

This is true as long as

$$k < \frac{n-1}{2} - \left(\frac{(n-2)^2}{12} - \frac{1}{12}\right)^{1/2} = n\left(\frac{1}{2} - \frac{1}{\sqrt{12}}\right) + O(1) \approx \frac{n}{4.7}.$$

In terms of labels of edges this implies that there is a line whose labels are all greater or equals to $(n-1)/2 - (((n-2)^2-1)/12)^{1/2} - 1$. This finally proves our main result.

THEOREM 2. *Let S be a set of n points in the plane such that no three points are collinear and no four are cocircular. Then there is a pair of points in S with the property that any circle through these two points encloses at least*

$$n\left(\frac{1}{2} - \frac{1}{\sqrt{12}}\right) + O(1) \approx \frac{n}{4.7}$$

points.

It is worthwhile to note that the lower bound of approximately $n/4.7$ does not meet the currently best upper bound of $\lceil n/4 \rceil - 1$. The authors of this paper believe that the upper bound (shown by a construction in [5]) is closer to the truth than their lower bound.

If S is a convex point set, then

$$\sum_{i=1}^{k} e_i(S) = kn.$$

In this case

$$\lambda_k = 2kn - \tfrac{3}{2}(k^2 + k).$$

Now $\lambda_k < \binom{n}{2}$ as long as k is smaller than $n/3$ which implies that there is a perpendicular bisector that contains no i-order edge, for $1 \leqslant i < \lceil n/3 \rceil - 1$. In other words there is always a pair of points such that any circle through the two points encloses at least $\lceil n/3 \rceil - 1$ of the points. This result was also obtained in [5] using a completely different method.

3. A LOWER BOUND FOR SEMISPACE

A *semispace* of a finite point set S in the plane is the intersection of S with a half-plane; it is called a *k-set* if its cardinality is k. This section proves a tight lower bound on the minimum number of semispaces of size k or less for k not exceeding $n/3$, n the cardinality of S. This lower bound is used in Section 2 to prove Theorem 2. To state the lower bound, we let $e_k(S)$ denote the number of k-sets of S.

THEOREM 3. *If no three points of a set S of n points in the plane are collinear, then*

$$\sum_{i=1}^{k} e_i(S) \geqslant 3\binom{k+1}{2}$$

and this lower bound is tight for $k \leqslant n/3$.

The remainder of this section presents a proof of Theorem 3. To see that the lower bound is tight consider a triangle and place one third of the points on the line segment that connects a corner with the center of gravity. Do this for every corner (see Figure 4). For $1 \leqslant i \leqslant n/3$, we have

$$e_i(S) = 3i \quad \text{and thus} \quad \sum_{i=1}^{k} e_i(S) = 3\binom{k+1}{2}.$$

To meet the assumption of Theorem 3 we perturb the points slightly so that no three points are collinear and the k-sets remain unchanged for $k \leqslant n/3$.

To prove the lower bound we consider a half-period of the circular sequence of S (see [3] or [2, Chap. 2]). This is the sequence of $\binom{n}{2} + 1$ permutations that represent the orthogonal projections of S onto a rotating line. The permuta-

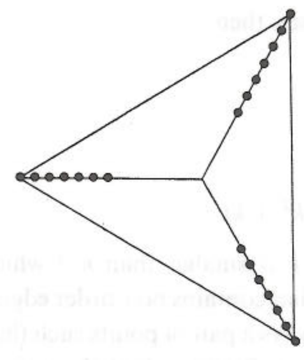

Fig. 4.

tions keep track of the different projections as the line rotates through π radians. If $(1, 2, \ldots, n)$ is the first permutation, then $(n, n-1, \ldots, 1)$ is the last one (see Figure 5). Furthermore, any two consecutive permutations differ by exactly one swap of an adjacent point pair, and any two points swap exactly once inside the half-period. (This assumes that no two point pairs lie on two parallel lines which is no loss of generality.) There is a one-to-one correspondence between the i-sets of S and the swaps that occur between columns i and $i+1$ and between $n-i$ and $n-i+1$. For $i \leqslant k$ we call these the *critical swaps*. Thus, $\Sigma_{i=1}^{k} e_i(S)$ is equal to the total number of critical swaps.

We will now argue about half-periods of circular sequences as defined above without considering geometric realizability. The advantage of doing so is that these are purely combinatorial objects and easier to manipulate locally than point sets. A lower bound proved for half-periods will also be a lower bound for the geometric problem. The reader who is unfamiliar with circular sequences may want to consult any one of the above mentioned sources.

To further classify the swaps, we let $X^{\mathscr{Y}}$ be the number of swaps that occur in region $\mathscr{Y} \in \{\mathscr{A}, \mathscr{B}, \mathscr{C}\}$ (see Figure 5). Thus, $\Sigma_{i=1}^{k} e_i(S) = X^{\mathscr{A}} + X^{\mathscr{C}}$.

It turns out that it is useful to push the classification even further. We write A for the points 1 through k, B for $k+1$ through $n-k$ and C for $n-k+1$ through n. $X^{\mathscr{Y}}_{P,Q}$ stands for the number of $P \mid Q$-swaps in region $\mathscr{Y}, P, Q \in \{A, B, C\}$, where a $P \mid Q$-*swap* changes an adjacent pair 'P to the left of Q' to 'Q to the left of P'. Since any two points swap only once within the half-period, there are no $B \mid A$-, $C \mid A$-, and $C \mid B$-swaps. Thus,

$$X^{\mathscr{Y}} = X^{\mathscr{Y}}_{A,A} + X^{\mathscr{Y}}_{A,B} + X^{\mathscr{Y}}_{A,C} + X^{\mathscr{Y}}_{B,B} + X^{\mathscr{Y}}_{B,C} + X^{\mathscr{Y}}_{C,C},$$

for $\mathscr{Y} \in \{\mathscr{A}, \mathscr{B}, \mathscr{C}\}$. We now show that we can assume w.l.o.g. that $X^{\mathscr{A}}_{A,A}$ and $X^{\mathscr{C}}_{C,C}$ vanish. We do this by manipulating the half-period that corresponds to

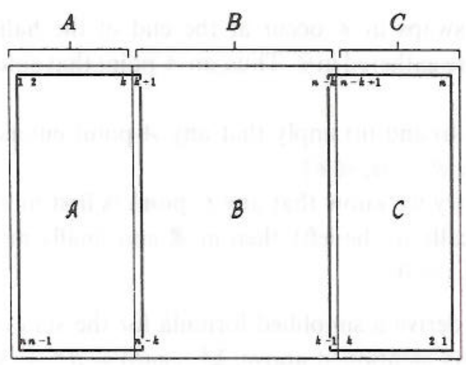

Fig. 5.

the point-set. The manipulations may or may not make sense in a geometric setting.

LEMMA 4. *For every half-period there is another half-period with the same number of critical swaps that has no $A\mid A$-swaps in $\mathscr{A}$ and no $C\mid C$-swaps in $\mathscr{C}$.*

Proof. Let α_1 and α_2 be the two A-points involved in the first $A\mid A$-swap within $\mathscr{A}$. We undo this swap and let α_1 follow the path of α_2 and α_2 follow the path of α_1 in the remainder of the half-period. This can be done since α_1 and α_2 are adjacent in the very first permutation (otherwise, other $A\mid A$-swaps have to be performed before the swap of α_1 and α_2) and are thus swapped with exactly the same points before we undo their swap.

We now get α_1 and α_2 adjacent in the last permutation of the half-period and we swap them then. This swap is within $\mathscr{C}$ which implies that the number of critical swaps is unaffected by this delaying transformation. The assertion follows since we can perform a sequence of such transformations eventually eliminating all $A\mid A$-swaps in $\mathscr{A}$, and similarly all $C\mid C$-swaps in $\mathscr{C}$.

By the same argument we can defer all $A\mid A$-swaps in $\mathscr{C}$ and all $C\mid C$-swaps in $\mathscr{A}$ to the end of the half-period without changing the number of critical swaps. Similarly, all $B\mid B$-swaps can be deferred to the end of the half-period without increasing the number of critical swaps. At the end, all B-points gather in $\mathscr{B}$ where any additional swaps do not affect our count at all. It is important for us that these transformations do not increase the number of critical swaps (they may decrease the number) and that they considerably simplify the pattern in which the swaps can occur. We review what these simplifications are.

(i) $X^{\mathscr{A}}_{A,A} = 0$ which implies that within $\mathscr{A}$ every A-point moves monotonically to the right.

(ii) All $A \mid A$ swaps in $\mathscr{C}$ occur at the end of the half-period when all A-points are gathered in $\mathscr{C}$. Thus, an A-point that enters $\mathscr{C}$ cannot leave it again.

(iii) Properties (i) and (ii) imply that any A-point enters and leaves $\mathscr{Y}$ at most once, $\mathscr{Y} \in \{\mathscr{A}, \mathscr{B}, \mathscr{C}\}$.

(iv) By symmetry we know that any C-point is first in $\mathscr{C}$ (where it moves monotonically to the left), then in $\mathscr{B}$, and finally in $\mathscr{A}$.

(v) $X^{\mathscr{A}}_{B,B} = X^{\mathscr{C}}_{B,B} = 0$

We use this to derive a simplified formula for the sum of the numbers of i-sets, $1 \leqslant i \leqslant k$. As mentioned above, $\Sigma_{i=1}^{k} e_i(S) = X^{\mathscr{A}} + X^{\mathscr{C}}$, and we have

$$X^{\mathscr{A}} = X^{\mathscr{A}}_{A,B} + X^{\mathscr{A}}_{A,C} + X^{\mathscr{A}}_{B,C} + X^{\mathscr{A}}_{C,C}$$

and

$$X^{\mathscr{C}} = X^{\mathscr{C}}_{A,A} + X^{\mathscr{C}}_{A,B} + X^{\mathscr{C}}_{A,C} + X^{\mathscr{C}}_{B,C}.$$

Clearly $X^{\mathscr{A}}_{A,B} + X^{\mathscr{A}}_{A,C} + X^{\mathscr{C}}_{A,B} + X^{\mathscr{C}}_{A,C} = k(k+1) = 2\binom{k+1}{2}$ because this counts for each A-point the number of right moves it takes to move from its initial to its final position, not counting the moves within $\mathscr{B}$. Symmetrically, $X^{\mathscr{C}}_{B,C} + X^{\mathscr{C}}_{A,C} + X^{\mathscr{A}}_{B,C} + X^{\mathscr{A}}_{A,C} = 2\binom{k+1}{2}$. Thus

$$\sum_{i=1}^{k} e_i(S) = 4\binom{k+1}{2} - X^{\mathscr{A}}_{A,C} - X^{\mathscr{C}}_{A,C} + X^{\mathscr{A}}_{C,C} + X^{\mathscr{C}}_{A,A}.$$

The end of the proof of Theorem 3 is near. To finish it we just need to show that $X^{\mathscr{A}}_{A,C} + X^{\mathscr{C}}_{A,C} - X^{\mathscr{A}}_{C,C} - X^{\mathscr{C}}_{A,A} \leqslant \binom{k+1}{2}$. To simplify the notation we define $X_{A,C} = X^{\mathscr{A}}_{A,C} + X^{\mathscr{C}}_{A,C}$, $X_{C,C} = X^{\mathscr{A}}_{C,C}$ and $X_{A,A} = X^{\mathscr{C}}_{A,A}$, and we set out to prove $X_{A,C} - X_{C,C} - X_{A,A} \leqslant \binom{k+1}{2}$.

We label the A-points as $A_1, A_2, \ldots, A_k$ such that A_i is the ith A-point entering $\mathscr{C}$. Similarly, the C-points are labelled such that C_i is the ith C-point entering $\mathscr{A}$. For each A_i we define $a_i^{\mathscr{C}}$ as the number of C-points that are in $\mathscr{C}$ as A_i is about to enter. We let $a_i^{\mathscr{A}}$ be the number of A-points in $\mathscr{A}$ at this moment. The contribution of A_i to $X_{A,C} - X_{C,C} - X_{A,A}$ is at most $a_i^{\mathscr{C}} - a_i^{\mathscr{A}}$ since it cannot swap with more than $a_i^{\mathscr{C}}$ C-points within $\mathscr{C}$ and it has to swap with at least $a_i^{\mathscr{A}}$ A-points within $\mathscr{C}$ later. In order to avoid double counting we consider any $A \mid C$-swap in $\mathscr{A}$ that A_i is involved in as the contribution of the C-point. Similarly, we define $c_i^{\mathscr{A}}$ as the number of A-points in $\mathscr{A}$ when C_i is about to enter $\mathscr{A}$ and $c_i^{\mathscr{C}}$ as the number of C-points left in $\mathscr{C}$ at this moment. The contribution of C_i to $X_{A,C} - X_{C,C} - X_{A,A}$ is at most $c_i^{\mathscr{A}} - c_i^{\mathscr{C}}$. We therefore have

$$X_{A,C} - X_{C,C} - X_{A,A} \leqslant \sum_{i=1}^{k} (a_i^{\mathscr{C}} - a_i^{\mathscr{A}} + c_i^{\mathscr{A}} - c_i^{\mathscr{C}}).$$

The trick that we use here is that we account for the combined contribution of A_i and C_i, for every i. If A_i enters $\mathscr{C}$ before C_i enters $\mathscr{A}$, then $a_i^{\mathscr{A}} \geqslant c_i^{\mathscr{A}}$ which implies $a_i^{\mathscr{C}} - a_i^{\mathscr{A}} + c_i^{\mathscr{A}} - c_i^{\mathscr{C}} \leqslant a_i^{\mathscr{C}}$. In the other case, that is, when C_i enters $\mathscr{A}$ before A_i enters $\mathscr{C}$, we have $c_i^{\mathscr{C}} \geqslant a_i^{\mathscr{C}}$ and thus $a_i^{\mathscr{C}} - a_i^{\mathscr{A}} + c_i^{\mathscr{A}} - c_i^{\mathscr{C}} \leqslant c_i^{\mathscr{A}}$. But now, $a_i^{\mathscr{C}} \leqslant k + 1 - i$ since $i - 1$ positions in $\mathscr{C}$ are already occupied by A_1 through A_{i-1}, and similarly $c_i^{\mathscr{A}} \leqslant k + 1 - i$. This yields

$$X_{A,C} - X_{C,C} - X_{A,A} \leqslant \sum_{i=1}^{k} (k + 1 - i) = \binom{k+1}{2}$$

which completes the proof of Theorem 3.

4. Discussion

In this paper we prove that every set of n points in the plane has two points such that any circle through the two points encloses at least $c \cdot n$ of the points, for $c \approx 1/(4.7)$. The best upper bound on the maximum c is $1/4$ and it is believed to be tight.

The lower bound on the minimum number of i-sets, for $1 \leqslant i \leqslant n/3$, of a set of n points in the plane (Theorem 3 in Section 3) has also applications to k-order Voronoi diagrams. It is known that the kth-order Voronoi diagram of a set S of n points (no three collinear and no four cocircular) has

$$R = (2k - 1)n - (k^2 - 1) - \sum_{i=1}^{k-1} e_i(S)$$

regions, $3R - 3 - e_k(S)$ edges, and $2R - 2 - e_k(S)$ vertices. Theorem 3 thus implies that for $k - 1 \leqslant n/3$ we have

$$R \leqslant (2k - 1)n - \frac{5k^2}{2} + \frac{3k}{2} + 1$$

regions and that this tight. It is rather straightforward to prove that $e_k(S) \geqslant 2k + 1$ which now implies that there are at most $(6k - 3) - 15k^2/2 + 5k/2 - 1$ edges and at most $(4k - 2) - 5k^2 + k - 1$ vertices, if $k - 1 \leqslant n/3$. These bounds are probably not tight since $e_k(S)$ and $\Sigma_{i=1}^{k-1} e_i(S)$ are not minimized by the same S.

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