Spatial Triangulations with Dihedral Angle Conditions

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Abstract

We study three types of spatial triangulations: Delaunay triangulations, triangulations with non-obtuse dihedral angles, and KJ-triangulations. The latter satisfy a certain angle condition useful for finite element approximation. We show that the condition for Delaunay triangulations is incomparable with the other two conditions, and that triangulations with non-obtuse dihedral angles are necessarily also KJ-triangulations. These relationships are in sharp contrast to the ones in the planar case.

1 Introduction

There are two reasons why in the plane Delaunay triangulations are popular for finite element approximation:

(i) for every interior edge the sum of the two opposite angles is at most π, and

(ii) for a finite set of points, the Delaunay triangulation maximizes the minimum angle over all triangulations of the point set.

Property (i) implies L_∞ stability (see Strang and Fix [5]), while convergence is guaranteed if small angles are avoided. We refer to [4, 2] for background information on Delaunay triangulations.

The situation is quite different in three dimensions. There are finite point sets so that the Delaunay triangulation maximizes neither the minimum solid angle, nor the minimum dihedral angle, nor the minimum (two-dimensional) face angle. It seems that, unlike in two dimensions, Delaunay triangulations in three dimensions do not distinguish themselves from other triangulations through any angle criterion. We add evidence to this view by showing that the sphere condition for Delaunay triangulations is incomparable with two natural conditions on the dihedral angles of triangulations. The first condition is that all dihedral angles be non-obtuse. The second will be stated in Section 2 and implies L_∞ stability for elliptic gradient equations in three dimensions, as recently shown in [3].

This paper is organized as follows. Section 2 introduces the formal definitions and states the main results of this paper. Section 3 reformulates the angle criterion given by Kerkhoven and Jerome [3]. In Section 4 we show that the sphere condition for Delaunay triangulations is incomparable with this condition, and in Section 5 we show it is incomparable with the requirement of having only non-obtuse dihedral angles. Section 6 studies the relationship between dihedral angles and face angles. Finally, Section 7 discusses a few open problems motivated by the results of this paper.

2 Definitions and Results

A *triangulation* is a cell complex consisting of relatively open pairwise disjoint simplices. By virtue of being a cell complex it contains the faces of all its simplices; these are lower-dimensional simplices.

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A triangulation *of a finite point set* S has the additional properties that the union of its simplices is the convex hull of S and that S is the set of 0-dimensional simplices. In three-dimensional space, the simplices are 3-dimensional (tetrahedra), 2-dimensional (triangles), 1-dimensional (edges) and 0-dimensional (vertices).

In this paper we consider three different kinds of triangulations distinguished by their geometric properties.

Delaunay triangulations. The edge ab defined by points a and b of a finite point set S is said to be *Delaunay* if there is a sphere through a and b so that all other points of S lie outside the sphere. Similarly, a two-dimensional face with vertices $a_1, a_2, \ldots, a_j$ from S is *Delaunay* if there is a sphere that contains a_1 through a_j and all other points of S lie outside the sphere.

If no five points of S are cospherical and S does not lie on a plane then each Delaunay face is a triangle and the collection of Delaunay edges and triangles defines a unique triangulation, the *Delaunay triangulation* of S. Otherwise, the Delaunay edges and faces define a cell complex whose cells are convex polytopes but not necessarily tetrahedra. This cell complex can be refined to a triangulation by decomposing non-triangular faces into triangles and non-tetrahedral cells into tetrahedra. This can be done with semi-Delaunay edges and triangles, where we call a non-Delaunay edge or triangle *semi-Delaunay* if there is a sphere through its vertices so that all other points of S lie on or outside the sphere. We call such a refinement a *Delaunay triangulation.*

In the literature, Delaunay triangulations that also contain semi-Delaunay edges or triangles are often called *completions* of the Delaunay cell complex. In this paper we prefer to give up uniqueness in exchange for always getting a triangulation rather than a possibly more general cell complex. We refer to Delaunay [1] where Delaunay triangulations are introduced and to [4, 2] for background information.

KJ-triangulations. We call a triangulation $\mathcal{T}$ of S a *KJ-triangulation* if it satisfies the following condition developed in Kerkhoven and Jerome [3]. It is expressed in terms of edges and dihedral angles.

For a tetrahedron $abcd$ let $|abcd|$ be its volume and for $x \in \{a, b, c, d\}$ let h_x be the distance of vertex x from the plane spanned by the opposite triangle. Let u_x be the inward directed normal vector of this triangle with length $1/h_x$ (see Figure 2.1). Now define

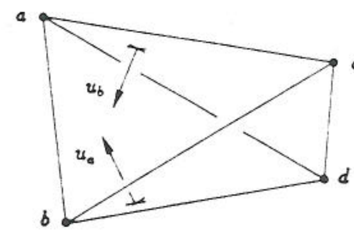

Figure 2.1: Tetrahedron with inward directed normal vectors.

$$f(ab, cd) = \langle u_a, u_b \rangle |abcd|$$

and

$$f(ab) = \sum_{cd} f(ab, cd),$$

where the sum is taken over all edges cd so that $abcd$ is a tetrahedron of $\mathcal{T}$. The condition developed in [3] is that $f(ab) \leq 0$ for each edge ab of $\mathcal{T}$.

Non-obtuse triangulations. A dihedral angle is *obtuse* if it exceeds the right angle, that is, $\frac{\pi}{2}$. A triangulation $\mathcal{T}$ of S is *non-obtuse* if all dihedral angles of its tetrahedra are non-obtuse.

The main result of this paper is the relationship between the three types of triangulations expressed by the following theorem.

Theorem 2.1 (1) Every non-obtuse triangulation is KJ but there are obtuse KJ-triangulations.

(2) There are non-obtuse triangulations (and therefore KJ-triangulations) that are not Delaunay and there are Delaunay triangulations that are not KJ (and therefore obtuse).

In other words, the non-obtuse dihedral angle condition and the condition of Kerkhoven and Jerome are incomparable to the sphere condition for Delaunay triangulations, and the non-obtuse dihedral angle condition is strictly stronger than the condition of Kerkhoven and Jerome. This is in sharp contrast to the planar case where Delaunay triangulations and KJ-triangulations are equivalent notions and where every non-obtuse triangulation is also Delaunay.

In addition to Theorem 2.1 we also show that the condition for non-obtuse dihedral angles is strictly stronger than for non-obtuse face angles.

3 KJ-Triangulations

We rewrite the KJ-condition presented in the previous section in terms of cotangents of dihedral angles. This will lead to a slightly more intuitive formulation.

Lemma 3.1 $f(ab, cd) = -\frac{1}{6}|cd| \cot\phi$, where $|cd|$ is the length of the edge cd and ϕ is the dihedral angle at cd inside the tetrahedron $abcd$.

Proof. If ψ is the angle between the vectors u_a and u_b then $\psi = \pi - \phi$. Therefore, $h_a h_b \langle u_a, u_b \rangle = \cos\psi = -\cos\phi$. This implies

$$f(ab, cd) = -\frac{|abcd|}{h_a h_b} \cos\phi.$$

Notice that $h_b = \frac{2|bcd|}{|cd|} \sin\phi$, where $|bcd|$ is the area of the triangle bcd, because $\frac{2|bcd|}{|cd|}$ is the height of triangle bcd with base cd (see Figure 3.1). This gives

$$f(ab, cd) = -\frac{\cos\phi}{\sin\phi} \frac{|abcd||cd|}{2|bcd|h_a} = -\frac{1}{6}|cd| \cot\phi.$$

□

With this lemma we can now rewrite the KJ-condition as

$$-6f(ab) = \sum_{cd} |cd| \cot\phi \geq 0$$

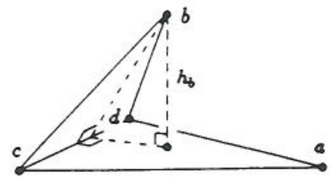

Figure 3.1: Reinterpreting the KJ-condition.

for all edges ab.

Remarks. (1) Recall that the cotangent of an angle ϕ is positive if $0 < \phi < \frac{\pi}{2}$, zero if $\phi = \frac{\pi}{2}$, and negative if $\frac{\pi}{2} < \phi < \pi$. This implies that the KJ-conditions is satisfied if each dihedral angle is non-obtuse which thus proves half of Theorem 2.1 (1).

(2) The KJ-condition as formulated in the previous section can be extended to arbitrary dimensions k; just replace volume by k-dimensional measure and take the sum over all k-dimensional simplices that have ab as an edge. This is the form proved in [3]. We see that it remains a condition on the edges of the triangulation. For example, in two dimensions, the sum is over the (at most) two incident triangles. Using the formulation with cotangents we get

$$f(ab) = -\frac{1}{2}(\cot\alpha + \cot\beta),$$

where α and β are the angles opposite to ab inside the two triangles. Now, $f(ab) \leq 0$ is equivalent to $\alpha + \beta \leq \pi$ which is exactly the Delaunay condition in two dimensions.

4 Delaunay Triangulations and KJ-Triangulations

We give two constructions to prove part of Theorem 2.1 (2), a Delaunay triangulation that violates the KJ-condition, and a KJ-triangulation that is not Delaunay.

A triangulation that is Delaunay but not KJ. Refer to the left part of Figure 4.1. Points a and

b are the north- and south-pole of a sphere σ and form the first two points of the five point example. Points c, d and e are the vertices of an equilateral triangle in the equator plane so that all three points lie outside the equator, and the three edges properly intersect the equator circle. The Delaunay triangulation of the five points consists of the tetrahedra $abxy$, with $xy \in \{cd, de, ec\}$. However, all opposite dihedral angles of ab are obtuse which implies that ab cannot be edge of any KJ-triangulation of the points.

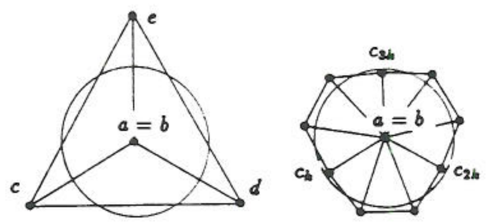

Figure 4.1: The top view of a Delaunay triangulation of five points that is not KJ is shown to the left. The top view of a KJ-triangulation that is not Delaunay is shown to the right.

A triangulation that is KJ but not Delaunay. The construction is illustrated in the right part of Figure 4.1. Take a and b as north- and south-pole of a sphere σ, as before, and for $k \geq 3$ choose a regular $3k$-gon with vertices $c_1, c_2, \ldots, c_{3k}, c_{3k+1} = c_1$ in the equator plane. The $3k$-gon is such that it contains the equator circle and the distance between the circle and each edge is equal to some $\delta > 0$. We choose δ small enough so that the line segment $c_{i-1}c_{i+1}$ properly crosses the equator circle and the midpoint lies at distance $\epsilon_0 > 0$ inside the circle. Next, we move three of the $3k$ vertices towards the center of the equator circle, namely points c_k, c_{2k} and c_{3k}, until they lie at distance $0 < \epsilon_1 < \epsilon_0$ inside the circle. Note that after moving the three points the $3k$-gon is still convex. The triangulation that we consider is formed by the tetrahedra abc_ic_{i+1}, for $1 \leq i \leq 3k$.

Notice that ab cannot be edge of any Delaunay triangulation because every sphere through a and b encloses at least one of the points c_k, c_{2k} and c_{3k}. It thus remains to show that the triangulation satisfies the KJ-condition. The only obtuse dihedral angles in the triangulation are the ones at the edges $c_{k-1}c_k$, c_kc_{k+1}, $c_{2k-1}c_{2k}$, $c_{2k}c_{2k+1}$, $c_{3k-1}c_{3k}$ and $c_{3k}c_{3k+1}$. Hence, the only edge that could possibly violate the KJ-condition is ab. However, the absolute contribution of these six edges to $f(ab)$ can be made arbitrarily small by choosing ϵ_0 small. Thus, this positive term is easily compensated by the acute angles at the other edges of the $3k$-gon.

Remark. The triangulation just described is also an example of an obtuse KJ-triangulation thus proving the remaining half of Theorem 2.1 (1).

5 A Non-obtuse Triangulations That Is Not Delaunay

We finish the proof of Theorem 2.1 with the construction of a non-obtuse triangulation that fails to be Delaunay. As a side result we get a non-obtuse triangulation that is *circumscribable* (that is, its vertices lie on a common sphere) and contains an arbitrary number of vertices. This is in contrast to the planar case where five or more points on a circle cannot be triangulated without obtuse angles.

We start with the construction of a non-obtuse triangulation whose $n \geq 5$ vertices lie on a sphere σ; it is illustrated in Figure 5.1.

(i) Choose $\epsilon > 0$ small enough and place $n-2$ points $c_1, c_2, \ldots, c_{n-2}$ on the equator so that the distance between c_i and c_{i+1} is ϵ.

(ii) For each edge c_ic_{i+1} let s_i be the open slab of points between the planes through c_i and c_{i+1} normal to the edge. The intersection of all s_i is a vertical cylinder which contains the north- and the south-pole of σ.

(iii) We choose points a and b on σ so that ab is parallel to the pole-axis, ab is contained in all s_i and is sufficiently far away from the c_i. We

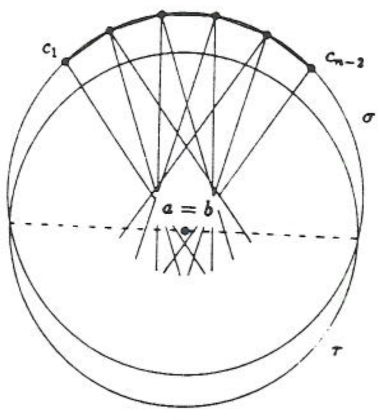

Figure 5.1: A set of n points on a sphere that admits a non-obtuse triangulation.

can assume that ab is arbitrarily close to the intersection of the plane through c_1 and that through c_{n-2}.

(iv) The tetrahedra are of the form abc_ic_{i+1} for $1 \leq i \leq n-3$.

The dihedral angles at ab are clearly non-obtuse. The dihedral angle at c_ic_{i+1} is non-obtuse because the line through c_i and c_{i+1} is normal to ab and disjoint from the sphere τ whose north-pole is a and whose south-pole is b. Finally, the dihedral angles at ac_i and bc_i are non-obtuse because in the orthogonal projection of abc_ic_{i+1} onto a plane parallel to ab and c_ic_{i+1} the two edges are normal to each other and intersect. Indeed all dihedral angles are acute, that is, smaller than $\frac{\pi}{2}$.

Finally, we move c_1 and c_{n-2} ever so slightly towards the center of σ. Because all dihedral angles were acute before the displacement of c_1 and c_{n-2} this can be done without introducing obtuse angles. But now c_1c_{n-2} is a Delaunay edge by construction which shows that the non-obtuse triangulation is not Delaunay.

6 Dihedral and Face Angles

This section studies the relationship between dihedral angles and (two-dimensional) face angles. We will see that a triangulation has no obtuse dihedral angle only if it has no obtuse face angle. On the other hand, there are triangulations without obtuse face angles that have obtuse dihedral angles. This shows that the condition for non-obtuse dihedral angles is strictly stronger than for non-obtuse face angles. The investigation will also shed new light on the relationship between non-obtuse triangulations and Delaunay triangulations. Everything is based on the following lemma.

Lemma 6.1 A tetrahedron has an obtuse dihedral angle if it has an obtuse face angle.

Proof. Let $abcd$ be the tetrahedron and let the angle γ at point c in abc be obtuse. Now consider the orthogonal projection of point d onto the plane through abc; it must lie in the (closed) triangle abc because, otherwise, we have an obtuse dihedral angle between abc and another triangle of $abcd$. We claim that in this case the dihedral angle along the edge cd is at least as large as γ and therefore obtuse.

To see this assume that d does not orthogonally project onto c; if it did the dihedral angle at cd would be equal to $\gamma > \frac{\pi}{2}$. Let η be the plane normal to cd through point d and consider the line ℓ that is the intersection between η and the plane spanned by abc (see Figure 6.1). Assume first that the half-line

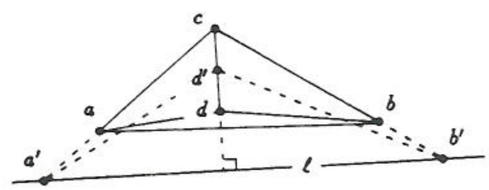

Figure 6.1: The angle at d' inside $a'b'd'$ is greater than the angle at c inside $a'b'c$.

$\vec{ca}$ that starts at c and passes through a intersects

ℓ and that $\vec{cb}$ intersects ℓ too. Define $a' = \ell \cap \vec{ca}$ and $b' = \ell \cap \vec{cb}$. Now rotate d about ℓ until it lies in the plane of abc; call its location d'. Because η is normal to cd the distance of d' from ℓ is smaller than the distance of c from ℓ. It follows that $a'b'd'$ is contained in $a'b'c$ and that therefore the angle at d' – which equals the dihedral angle at cd – is greater than the angle at c.

In the other case we assume that $\vec{ca}$ does not meet ℓ (see Figure 6.2). Define d' and b' as before and set

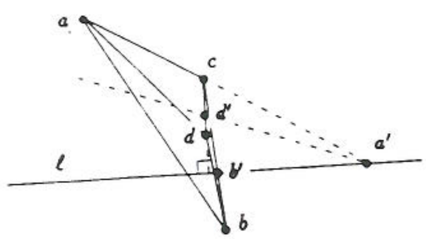

Figure 6.2: The angle at d' is necessarily obtuse.

$a' = \ell \cap \vec{ac}$, assuming that a' exists. The dihedral angle at cd is equal to the angle at d' defined by the line through d' and a' and the line through d' and b'. This angle is obtuse because a' and b' lie on the same side of the normal to ℓ through d' (see Figure 6.2).

Finally, if a' does not exist, that is, ac is parallel to ℓ, then the angle at d' is obtuse by the same reasoning. □

Remarks. (1) Let a and b be two antipodal points on a sphere σ, let c be inside σ and let d be anywhere. Lemma 6.1 implies that $abcd$ has an obtuse dihedral angle because the face angle at c inside abc is obtuse.

(2) Remark (1) implies an interesting fact about the relationship between non-obtuse triangulations and Delaunay triangulations. Let $\mathcal{T}$ be a non-obtuse triangulation. For each edge consider the unique sphere σ that contains a and b as antipodal points. No point c connected to both, a and b, can lie inside σ; otherwise, we have a contradiction to Lemma 6.1. In other words, all edges of $\mathcal{T}$ are locally Delaunay [2]. From what we have seen earlier, $\mathcal{T}$ is still not necessarily a Delaunay triangulation. This should be contrasted to the fact that $\mathcal{T}$ is necessarily a Delaunay triangulation if all its edges *and* triangles are locally Delaunay [2].

We conclude this section with a triangulation whose face angles are all acute but which has obtuse dihedral angles. The triangulation consists of a single tetrahedron whose vertices are the points $(1,0,-\epsilon)$, $(-1,0,-\epsilon)$, $(0,1,\epsilon)$, and $(0,-1,\epsilon)$, for some sufficiently small $\epsilon > 0$.

7 Conclusions and Open Problems

This paper compares three types of triangulations of finite point sets in three-dimensional space: Delaunay triangulations, KJ-triangulations and non-obtuse triangulations. The main result is that the condition for non-obtuse triangulations is strictly stronger than for KJ-triangulations and that the condition for Delaunay triangulations is incomparable with the other two. In addition, this paper shows that all dihedral angles can be non-obtuse only if all face angles are non-obtuse.

Of course, there are point sets that do not admit non-obtuse triangulations, unless extra points can be added. This suggests the following question.

> What is the minimum $g(n)$ so that for any set P of n points in three-dimensional space there is a set $Q \supseteq P$ of at most $g(n)$ points so that Q admits a non-obtuse triangulation?

It is fairly easy to show $g(n) \leq n^3$ as follows. Through each point of P draw the three planes parallel to the coordinate planes and take Q as the set of intersection points between the planes. Each bounded box can be decomposed into six non-obtuse tetrahedra by choosing a diagonal and taking the tetrahedra defined by this diagonal and the six edges of the box that do not share an endpoint with the diagonal. If we choose the diagonals consistently then the tetrahedra define a triangulation. The hope is that $g(n)$ is much smaller than cubic in n.

The same problem can also be asked for KJ-triangulations, for triangulations with acute dihedral angles only, and for triangulations with no dihedral angles exceeding α, for some $\frac{\pi}{3} < \alpha < \frac{\pi}{2}$. For the latter two problems it is not clear whether $g(n)$ exists.

On a more global level, it is interesting to study how much more general KJ-triangulations are than non-obtuse triangulations.

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