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COVERING CONVEX SETS WITH NON-OVERLAPPING POLYGONS*

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We prove that given $n \geq 3$ convex, compact, and pairwise disjoint sets in the plane, they may be covered with n non-overlapping convex polygons with a total of not more than $6n-9$ sides, and with not more than $3n-6$ distinct slopes. Furthermore, we construct sets that require $6n-9$ sides and $3n-6$ slopes for $n \geq 3$. The upper bound on the number of slopes implies a new bound on a recently studied transversal problem.

1. Introduction

Consider a collection of n convex, compact, and pairwise disjoint sets labeled from 1 through n in the plane. We wish to cover each set i with a convex polygon a_i, such that no two polygons overlap. Here, a *convex polygon* is defined as the bounded intersection of a finite number of closed half-planes. A polygon with k sides is also called a *k-gon*. Wenger [5] shows that the polygons can be chosen such that not more than $12n+12$ sides realizing not more than $6n+6$ distinct slopes are required. In this paper we improve the bounds by showing that $6n-9$ sides and $3n-6$ distinct slopes suffice, that is, $n \geq 3$ convex sets may be covered by a set of n disjoint k_i-gons, $1 \leq i \leq n$, where

$$\sum_{i=1}^{n} k_i \leq 6n-9.$$

Furthermore, for $n \geq 3$, we construct sets that require $6n-9$ sides and $3n-6$ distinct slopes to cover. Thus, our bounds on the number of sides and slopes are tight.

The organization of this paper is as follows. In Section 2 we demonstrate lower bounds on the number of sides and slopes needed. In Section 3, we describe and

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analyze a construction of polygons which proves the upper bounds. Finally, we apply the result to a transversal problem and a triangulation problem in Section 4.

2. A lower bound construction

For each $n \geqslant 3$ there is a collection of n compact, convex, and pairwise disjoint sets in the plane that simultaneously requires the maximum number of sides and the maximum number of slopes to be covered. These sets are described in the proof of the following theorem which states the lower bound.

Theorem 1. *To cover $n \geqslant 3$ compact, convex, and pairwise disjoint sets by disjoint convex polygons, one per set, may require $6n-9$ sides and $3n-6$ distinct slopes.*

Proof. Construct an equilateral triangle Δabc. Now, construct an equilateral triangle $\Delta a'b'c'$, which is a $\frac{1}{3}$ scale copy of Δabc with the same center, but rotated by π. Next connect the vertices of Δabc and $\Delta a'b'c$ to form a graph as shown in Fig. 1. (The triangles are in broken lines; the graph is solid.) Recursively repeat the process by letting the old $\Delta a'b'c'$ be the new Δabc. We finish the construction by completing the inner-most part with one of the three constructions shown in Fig. 2 depending on whether n modulo 3 is 0 (Fig. 2a), 1 (Fig. 2b), or 2 (Fig. 2c). Topologically, we can view the construction as what we see if we wrap a hexagonal grid around the inside of a hollow cylinder, and then look down the cylinder axis. Now perturb each vertex slightly so that each edge of the graph has a distinct slope, and shrink each face by some small amount ε to form n disjoint convex sets.

Consider the construction for n a multiple of 3. The three outer-most faces require pentagons to cover; the three inner-most faces require quadrilaterals to

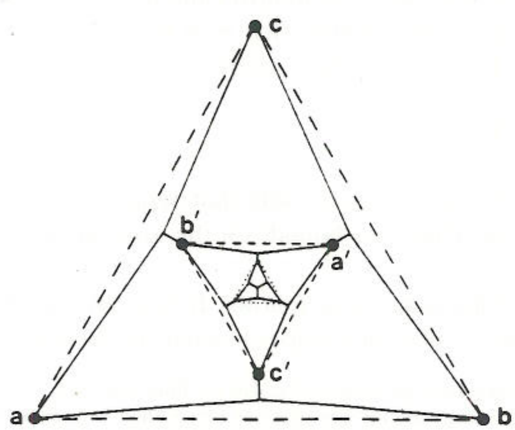

Fig. 1. A lower bound example.

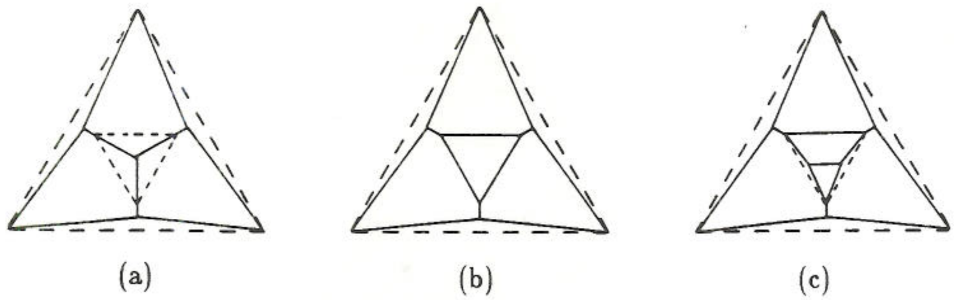

Fig. 2. Completing the lower bound construction.

cover; all other faces require hexagons. Thus at least $6 \cdot (n-6) + 5 \cdot 3 + 4 \cdot 3 = 6n - 9$ sides are required for a covering. If we disregard the three outer-most sides, then each side of the covering polygons is parallel to at most one other side, provided ε is small enough. This implies that the construction requires at least $(6n - 9 - 3)/2 = 3n - 6$ distinct slopes. Similar inspection of the other two cases shows that they also require $6n - 9$ sides and $3n - 6$ slopes. □

Note that the sets constructed in the above proof can in fact be covered by polygons with a total number of $6n - 9$ sides and $3n - 6$ slopes. This is because the slopes of the three outer-most sides can be chosen to be equal to three other slopes. This is not quite true only if $n = 3$. In this case we can find three covering polygons with 9 sides and also polygons with 3 slopes, but there are no three polygons that achieve both bounds simultaneously (see Fig. 3).

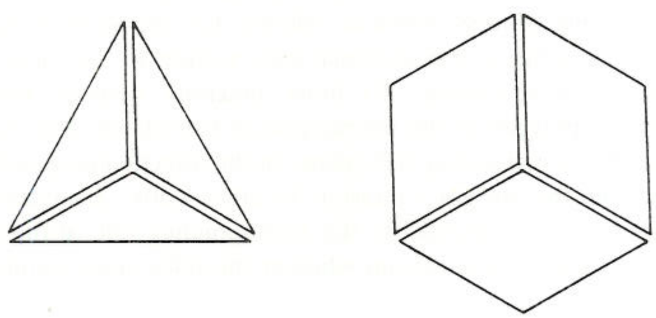

Fig. 3. Covering 3 sets with 9 sides or 3 slopes.

3. Area maximal polygons

We begin our upper bound construction by circumscribing polygons around the convex sets, such that all sides have distinct slopes. Since the objects are convex and compact this is always possible such that no two polygons intersect. The

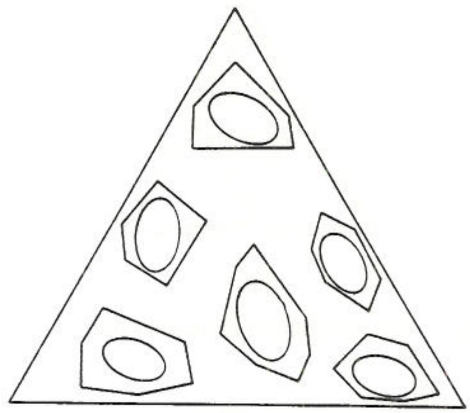

Fig. 4. Circumscribing polygons around the sets.

reason for choosing pairwise non-parallel sides is technical. At this stage we are neither concerned about the number of sides of the polygons nor about the number of slopes. Let A be the set of polygons $\{a_1, a_2, \ldots, a_n\}$. We then circumscribe a triangle around all the polygons. Let $\varepsilon > 0$ be the minimum distance between a polygon's boundary and its corresponding convex set. We will use ε in the last stage of the construction. Fig. 4 shows an example of the construction at the current stage.

We now grow the polygons in A so that they maximize their area. The growing process will create polygons with overlapping boundaries but disjoint interiors. This deficiency will eventually be remedied by shrinking each polygon by a small amount. The growing process works as follows. Let the sides of each polygon move out until each polygon is of maximal area, subject to the constraint that no two polygons' interiors overlap. To more precisely describe the expansion process, consider a polygon as the intersection of half-planes. Moving out a side means to move the corresponding half-plane, in the direction perpendicular to the side and away from the polygon's interior. In other words, each side of the new expanded polygon remains parallel to the corresponding side of the unexpanded polygon. For our result it is irrelevant whether the sides move simultaneously or

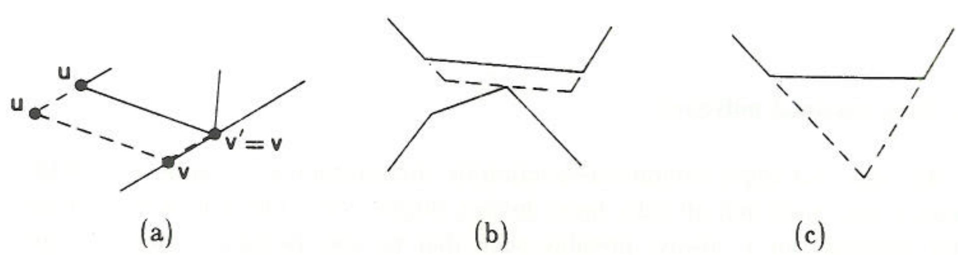

Fig. 5. Growing a polygon.

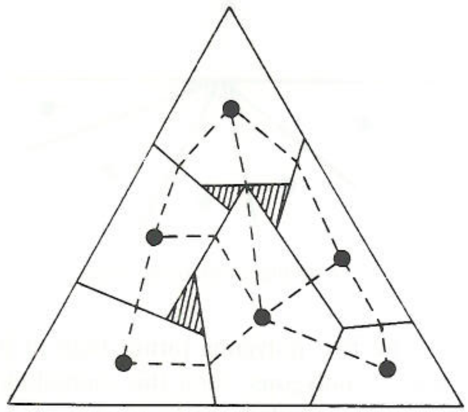

Fig. 6. Maximal polygons and their contact graph.

one after the other in an arbitrary order. Notice, however, that different schedules lead to different polygons.

There is one case where we have to add another side to a polygon, as shown in Fig. 5a. If the endpoint v of a side uv encounters the side of another polygon, we grow a new side $v'v$ at vertex v and continue moving uv outwards. The new side lies on the same line as the side that vertex v encoutered. Note that the expanding process is finite since a side is stopped forever if it is stopped. There are two conditions that will stop a side from moving further. First, the side touches another polygon's corner (see Fig. 5b), or, second, a side shrinks to a point and vanishes (see Fig. 5c).

We introduce a few definitions in order to analyze the polygons constructed as described above. We say that polygons a_i and a_j are *in contact* if a side of a_i intersects the boundary of a_j or, vice versa, a side of a_j intersects the boundary of a_i. We consider a side as a relatively open set, that is, it does not include its endpoints. Then a_i and a_j are not in contact if they share a single point which is a corner of both. We now construct the *contact graph* G of A. G contains a vertex for each polygon in A and an edge between any two vertices that correspond to polygons in contact. More formally, $G=(V, E)$ with $V=\{v_i \mid a_i \in A\}$ and $E=\{\{v_i, v_j\} \mid a_i \text{ contacts } a_j\}$. To avoid confusion, we refer to elements of the polygon's boundaries as sides and corners, and to elements of the contact graph as edges and vertices. Fig. 6 shows an example of a set of polygons and the corresponding contact graph. It is obtained by growing the polygons in Fig. 4.

Lemma 1. *The contact graph G is planar.*

Proof. We embed the contact graph in the plane as follows. We put each vertex, v_i, inside its corresponding polygon, a_i. Any two contacting polygons, a_i and a_j, share a point, p, on their boundary, so we can draw the corresponding edge straight from v_i to p and then straight from p to v_j. Whenever p is not unique we

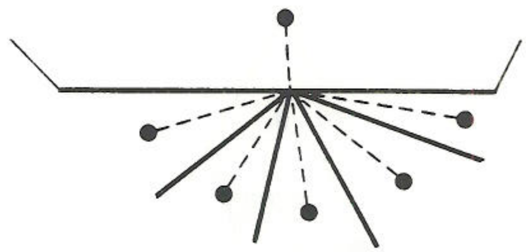

Fig. 7. Avoiding overlapping edges.

choose it on the sides of the two polygons rather than at a corner. If p is not common to any other pair of polygons, then this embedding of $\{v_i, v_j\}$ cannot intersect any other edge which is similarly embedded.

There, is however, one case in which we produce intersecting edges. Let p be a point of a side of a_i that is also the corner of at least two other polygons (see Fig. 7). If such a case occurs, then our embedding of the contact graph is not plane but we can still argue that the contact graph is planar (that is, it allows a plane embedding). Notice that the side of a_i that contains p also contains an open interval including p that does not intersect any other edges. Choose k points in this interval where k is the number of polygons which have a corner at p. We can now draw the k edges through these k points rather than through p. Nonintersecting edges are guaranteed if the order of the points in this interval matches the order of the polygons around point p. □

From now on when we talk about the contact graph, we mean the original embedding which may contain overlapping edges. To use planarity of the contact graph for proving an upper bound on the total number of sides, we have to relate edges with sides.

Lemma 2. *Every side of each polygon is crossed by at least one edge of the contact graph.*

Proof. Suppose a side is not crossed by an edge. Then the side does not touch another corner or side. In this case it can be moved further to increase the area of the polygon, thus the polygon is not maximal, a contradiction. □

Using Lemmas 1 and 2 we can prove that $6n-3$ sides and $3n$ slopes are sufficient to cover n convex sets by disjoint convex polygons. However, to prove tight upper bounds we need one more step in the construction. For this step it is essential to understand the structure of a hole in a maximal construction, where a *hole* is defined as a connected component of the complement of the union of polygons inside the outer triangle (see shaded areas in Fig. 6). Clearly, any hole is an open, bounded polygon.

Lemma 3. *The outer triangle of a maximal construction cannot intersect the boundary of a hole.*

Proof. Assume the outer triangle touches a hole and let $e_1, e_2, \ldots, e_k$ be the sides of the hole enumerated in counterclockwise order around the hole. Each e_i belongs to a side f_i of a polygon or of the outer triangle. This side properly contains e_i, for, otherwise, the polygon is not maximal. Let e_1 be a side whose one endpoint lies on the outer triangle. It follows that f_1 contains one endpoint of f_2, f_2 contains one endpoint of f_3, etc. This eventually contradicts since f_1 cannot contain an endpoint of f_k which is a side of the outer triangle. □

Note that it is straightforward to use the proof of Lemma 3 for showing that each hole is convex. This is, however, immaterial for proving the main result of this section. We now continue with the construction. Call a side a *spoke* if one of its endpoints (or both) lie on the outer triangle and the side is not colinear with a side of the other triangle. By Lemma 3, a spoke lies in the common boundary of two polygons. Grow this triangle continuously. As the triangle grows, the spokes are extended and the incident polygons are expanded. When two spokes meet, we let them stop there and start a new spoke that maintains convexity. Continue this process until the triangle disappears to infinity. When this process is done, we are left with a set of spokes that radiate outwards to infinity.

Reconstruct the contact graph. Lemmas 1 and 2 still hold for the new contact graph. Any new sides added are between two polygons and thus are crossed by edges of the contact graph. Call a vertex v_i of the contact graph *peripheral* if it lies on the unbounded face and let $d(v_i)$ denote the degree of v_i. By construction, v_i corresponds to an unbounded polygon with at most $d(v_i)$ sides.

Lemma 4. *For every peripheral vertex v_i of the contact graph for $n \geqslant 3$ polygons, we have $d(v_i) \geqslant 2$.*

Proof. Suppose $d(v_i) = 0$. By Lemma 1, the unbounded polygon a_i must have no sides, therefore all of its sides must have been removed with the outer triangle. Thus the polygon must cover the entire plane, a contradiction since there are at least two other polygons.

Suppose $d(v_i) = 1$. Then the unbounded polygon a_i has a single side l which is unbounded on both ends. Let l' and l'' be the two spokes adjacent to l and let a_j be the ploygon that contacts a_i. Since a_i contacts a_j only, l' and l'' must be sides of a_j. But a_j is convex and neither l' nor l'' may be parallel to l. This is because the construction started out with pairwise non-parallel sided and no new slopes are ever introduced. This also prevents the creation of two parallel sides which do not belong to the same line. Therefore, either l' or l'' must intersect l, a contradiction. □

By construction of the contact graph, $G=(V, E)$, each edge either crosses a corner touching a side or it crosses two collinear sides. We partition E accordingly, that is, $E=E_{cs} \cup E_{ss}$, where E_{cs} contains all edges of E that go through a corner and E_{ss} contains all edges that cross two sides. Using Lemmas 1 through 4 we can now prove tight upper bounds on the number of sides and slopes.

Theorem 2. *A collection of n compact, convex, and pairwise disjoint sets in the plane may be covered with n non-overlapping convex polygons with a total of not more than* $6n-9$ *sides. Furthermore no more than* $3n-6$ *distinct slopes are required.*

Proof. Let S be the set of all polygon sides and m be the number of distinct slopes. At most, we have one side in S for each edge in E_{cs}, and two sides in S for each edge in E_{ss}. Thus,

$$|S| \leq |E_{cs}| + 2|E_{ss}|.$$

Likewise, we have at most one slope for each edge in E_{cs} or E_{ss}, which implies

$$m \leq |E_{cs}| + |E_{ss}|.$$

Let k be the degree of the unbounded face of the contact graph, that is, the number of edges on the periphery. A planar graph with a face of degree k has no more than $3n-3-k$ edges. Thus, we have

$$|E_{cs}| + |E_{ss}| \leq 3|V| - 3 - k = 3n - 3 - k$$

since $|V|=n$ by definition of the contact graph. This implies

$$|S| \leq |E_{cs}| + 2|E_{ss}| \leq 2(|E_{cs}| + |E_{ss}|) \leq 2(3|V| - 3 - k) = 6n - 6 - 2k,$$

and

$$m \leq |E_{cs}| + |E_{ss}| \leq 3n - 3 - k.$$

Now we bound each of the k unbounded polygons by adding extra sides at least ε away from the corresponding sets. If we aim at minimizing the number of sides we choose one side per unbounded polygon. An exception occurs if one of the unbounded polygons is a half-plane, in which case we need yet another side to bound the polygon. The degree of the corresponding contact graph vertex must be at least 2, so we have counted the half-plane boundary as two sides. Use the credit for an extra side to bound the polygon. If our goal is to minimize the number of slopes, we choose the additional sides parallel to existing sides. This may force us to pick two sides for an unbounded polygon if this polygon is bounded only by two spokes.

Let S' be the new set of sides and m' the number of slopes. We have

$$|S'| \leq |S| + k = 6n - 6 - k \leq 6n - 9$$

and

$$m' = m \leqslant 3n - 3 - k \leqslant 3n - 6.$$

Shrink the polygons by ε to generate the final set of disjoint covering polygons. □

4. Applications to two combinatorial geometry problems

We apply our results to the problems of transversals and triangulations of a set of convex objects.

A Transversal Problem. The original motivation for studying the covering problem of this paper stems from a transversal problem defined for a finite collection of convex, compact, and pairwise disjoint sets in the plane. Let S be such a collection and let the sets be labeled from 1 through n. A line that cuts all sets is called a *transversal* of S. Clearly, a transversal intersects the sets in a well-defined order which can be expressed by a permutation of $(1, 2, \ldots, n)$ and by its reverse since a transversal is not directed. Such a pair of permutations induced by a transversal is called a *geometric permutation* of S.

In two papers Katchalski et al. [3, 4] study the maximum number of geometric permutations that can be realized by any collection of n convex, compact, and pairwise disjoint sets. They prove that $2n-2$ is a lower bound if $n \geqslant 4$, and that $\binom{n}{2}$ is an upper bound for this number. Wenger [5] reduces the transversal problem to the covering problem studied in this paper and proves that $6n+6$ is an upper bound. Our analysis (Theorem 2) improves this bound to $3n-6$. Finally, Edelsbrunner and Sharir [1] prove that $2n-2$, the lower bound established in [4], is also an upper bound and thus the answer to the extremal problem if $n \geqslant 4$.

Thus, it appears that the upper bound of [1] is strictly stronger than what can be obtained from Wenger's reduction together with our analysis of the covering problem. This is not really the case since the reduction is applicable to a more general extremal problem that also considers lines missing some of the sets. Define $\pi(S)$ as the smallest integer such that there are $\pi(S)$ permutations of S with the following property. If l is a directed line, then the sequence of sets met by l is a subsequence of one of the $\pi(S)$ permutations or its reverse. We are interested in

$$\pi(n) = \max\{\pi(S) \mid |S| = n\}.$$

Below we state the result which follows from Theorem 2. For completeness, we also indicate the main steps needed to prove that Theorem 2 implies the result.

Theorem 3. *$\pi(n) \leqslant 3n - 6$ if $n \geqslant 3$.*

The main idea of the proof is the construction of a set of lines, H, such that any two sets in S are separated by at least one line. To each line we assign its angle in $[0, \pi)$ and we assume without loss of generality that there is at least one line with angle 0. The $m \leq |H|$ angles cut the interval $[0, \pi)$ into m open intervals. For each interval, we can give a permutation such that a new line whose angle lies in this interval intersects the sets in a not necessarily consecutive subsequence of this permutation or its reverse. The construction of the permutation is straightforward: for every pair of sets, the line in H that separates the two sets decides which one of the two goes first.

Thus, the problem is now reduced to finding a set H with small angle set. To get such a set, we cover each set by a convex polygon (see Section 3), and for each side we add the line that contains it to H. For any two disjoint convex polygons in the plane there is at least one side whose extension to a line separates the polygons. This lemma proves that the set H thus constructed contains a separating line for every pair of sets. By Theorem 2, there is a covering by polygons with a total number of at most $3n-6$ different slopes which implies that the size of the angle set of H is at most $3n-6$ and this proves the result.

It is interesting to note that the construction of H given above is optimal. In fact, the lower bound example of Section 2 (see Fig. 1) shows n sets that require $3n-6$ lines to separate each pair. It is, however, not clear whether or not Theorem 3 is the best possible. Currently, the best lower bound for $\pi(n)$ is $2n-2$ which follows from the lower bound for the transversal problem mentioned in the introduction of this section.

A Triangulation Problem. Florian and Schmidt [2] recently considered the following problem which is related to the polygon covering problem studied in this paper.

> Decompose a given triangle that contains n convex and pairwise disjoint objects into triangles so that every triangle intersects at most one of the objects.

They show that $6n-5$ triangles are sufficient and that the multiplicative factor, 6, is the best possible. Our construction can be used to show a tight bound of $6n-11$ triangles.

We decompose the triangle into triangles as follows. Construct the covering polygons as described in Section 3. We will have a polygonal covering and contact graph as shown in Fig. 6. The results up until Lemma 3 hold. We do not grow the outer triangle. Now triangulate each polygon and each hole.

Each i-gon will require $i-2$ triangles. We will count the total number of sides and then subtract the number of polygons. Let V, E, and F be the vertices, edges, and bounded faces respectively of the contact graph and let k be the degree of the outer face of the contact graph. Then there are no more than $2|E|$ polygon sides corresponding to contact graph edges, and we have $k+3$ extra sides around the periphery. Since there are $2|V|$ polygons, we require no more than $2|E|+k+3-2|V|$ triangles to triangulate the polygons.

We now consider the hole triangulation. Each hole corresponds to a face of the contact graph. By Lemma 3, no hole may touch the outer triangle. By Lemma 5 (which we prove below), the number of sides of a hole is no more than the degree of the corresponding face. Furthermore, we need $2|E|-k-2|F|$ triangles to triangulate the holes, the $-k$ term follows from the fact that the sum of the face degrees is $2|E|-k$.

Altogether we need $4|E|-2(|V|+|F|)+3$ triangles. But $|V|+|F|=|E|+1$, and we have shown in Section 3 that $|E| \leqslant 3n-k-3$. Therefore the number of triangles is bounded by $6n-2k-5$. Since $k \geqslant 3$, we have the upper bound of $6n-11$.

Lemma 5. *The number of sides of a hole is no more than the degree of the surrounding face of the contact graph.*

Proof. The sides of the hole are sides of bounding polygons. Therefore each side of the hole must pass through an edge of the contact graph. Furthermore, since any two polygon sides passing through the same edge must be collinear, no more than one of these can be a side of the hole. □

One small technicality remains: the outer face may not be a simple path, that is, may contain an edge twice. In this case, each extra side also adds an extra edge to the outside hole that we threw out, so there is no net gain in requisite triangles.

Finally, we show the $6n-11$ bound is tight by triangulating perturbations of Figs 1 and 2. We perturb the constructions in Figs 1 and 2 such that the interior vertices are replaced with triangular holes. Perturbing the figure one creates $2n-5$ triangular holes. The polygons require $6n-6$ sides (the three outermost faces now require hexagons), therefore the polygons will require $4n-6$ triangles. Thus we need $6n-11$ triangles to cover the perturbed construction.

Theorem 4. *A triangle that contains n convex and pairwise disjoint objects can be decomposed into at most $6n-11$ triangles so that the interior of each triangle intersects at most one object. This bound is tight.*

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