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Tetrahedrizing Point Sets in Three Dimensions†

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This paper offers combinatorial results on extremum problems concerning the number of tetrahedra in a tetrahedrization of n points in general position in three dimensions, i.e. such that no four points are co-planar. It also presents an algorithm that in $O(n \log n)$ time constructs a tetrahedrization of a set of n points consisting of at most $3n-11$ tetrahedra.

1. Introduction

Tetrahedrizing a point set in three-dimensional Euclidean space is not only a natural generalization of the well-studied problem of triangulating a point set in two dimensions, but it is also central to a number of applications in numerical computing (Strang & Fix, 1973) and in solid modelling (Cavendish *et al.*, 1985). Indeed, both the solution of partial differential equations by the finite element method and the structural analysis of complex physical solids require the decomposition of a given spatial domain into elementary cells, which, in their simplest form, are tetrahedra. The problem is formulated as follows:

given a set P of n points in three dimensions, a *tetrahedrization* of P is a decomposition of the convex hull of P into (solid) tetrahedra, such that
(i) P contains the four vertices and no other points of each tetrahedron, and
(ii) the intersection of two tetrahedra is either empty or a face of each.

Here we use "face" in its general simplicial meaning; a *face* of a tetrahedron is the convex hull of some of its vertices, that is, it is the set of convex combinations of these vertices. A *facet* is the convex hull of three vertices of a tetrahedron.

Traditionally, in solid modelling applications desirable triangulations and tetrahedrizations are those which avoid thin and elongated cells. The Delaunay triangulation (Preparata & Shamos, 1985) and its three-dimensional counterpart are very attractive because they exhibit the above property. However, whereas in two dimensions any triangulation of any set of n points (Delaunay or otherwise) has $\Theta(n)$ cells, it has been noted (Preparata & Shamos, 1985; Klee, 1980; Seidel, 1982) that a Delaunay tetrahedrization may consist of $\Theta(n^2)$ cells. This fact, perhaps, has held back the investigation of this problem in the context of computational geometry.

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In this paper we undertake this study. To defer the study of degeneracies to a later stage, we assume that the points of P are in reasonably defined general position, that is, no four points are co-planar, or, with reference to Delaunay tetrahedrizations, no five points are co-spherical. It is important to note that degenerate point sets do not behave as well as non-degenerate point sets; Avis & ElGindy (1987) contains results to this effect. This paper is organized as follows. In section 2 we present several combinatorial results regarding tetrahedrizations. Let the *size* of a tetrahedrization denote the number of tetrahedra used. We give precise bounds on the size of a tetrahedrization of n points and present results on four naturally arising extremal problems. In section 3 we present a best-possible result on balanced partitioning of a set of points internal to a tetrahedron. Such partitioning, achievable in linear time, is central to a tetrahedrization algorithm—called Stepwise Refinement—developed in section 4. It produces a tetrahedrization of size at most $3n-11$ in $O(n \log n)$ time, but may create thin and elongated cells. Finally, in section 5 we discuss two additional tetrahedrization algorithms. One constructs the Delaunay tetrahedrization and runs in time $O(n^2)$. The second—called Greedy Peeling—constructs a tetrahedrization with at most $5n-19$ cells and runs in time $O(n^{3/2} \log n)$. Some interesting open questions—both algorithmic and combinatorial—are presented in section 6.

2. Combinatorial Facts about Tetrahedrizations

We start with Euler's formula for three-dimensional complexes, $n-e+f-c=0$, where c is the number of cells, f is the number of facets (boundaries between cells), e is the number of edges (boundaries between facets), and n is the number of vertices. This formula is a corollary of more general theorems in homology theory (see for example, Greenberg (1967)); an elementary proof can be found in Hopf (1953). In a tetrahedrization of P, the set of vertices is precisely P. All cells are tetrahedra, except the unbounded cell, and all facets are triangles. Letting t be the size of the tetrahedrization, that is, the number of internal tetrahedra, we have $n-e+f-t=1$. We will henceforth, occasionally refer to the internal tetrahedra of a tetrahedrization as its *cells*, for ease of expression. Furthermore, we will refer to the vertices, edges and facets on the boundary of the convex hull of P as the *hull vertices, hull edges* and *hull facets*, respectively. Other vertices, edges and facets will be said to be *interior*. Throughout the paper, we let n' be the number of hull vertices, and we define $n''=n-n'$, the number of interior vertices.

The hull vertices and hull edges form a maximal planar graph. Thus, there are $3n'-6$ hull edges and $2n'-4$ hull facets. Every interior facet belongs to two cells and every cell has four facets. Thus $4t+(2n'-4)=2f$, and we can eliminate f from the earlier formula to obtain the fundamental combinatorial relationship of tetrahedrizations.

LEMMA 2.1. *For any tetrahedrization of a set of n points in general position, with n' hull points and n'' interior points, we have*

$$t=e-n-n'+3=e-2n'-n''+3.$$

In particular, given fixed values for n' and n'', such as for a fixed point set, we get one additional cell for each additional edge used. There are at most $\binom{n}{2}$ edges altogether, and always at least the $3n'-6$ hull edges. In addition, every interior vertex belongs to at least four edges, so there are at least $2n''$ edges involving interior vertices. Hence we get trivial bounds

$$n-3\leq t\leq\binom{n-1}{2}-n'+2.$$

Given a point set P, let $\tau(P)$ and $T(P)$ be the minimum and maximum size of its tetrahedrizations. This suggests four types of extremal questions. Let $P_{n,n'}$ be the collection of point sets consisting of n points, of which n' lie on the boundary of the convex hull. Let

$$T(n, n')=\max_{P\in P_{n,n'}}\{T(P)\},\ \tau(n, n')=\min_{P\in P_{n,n'}}\{\tau(P)\},$$

$$\rho(n, n')=\max_{P\in P_{n,n'}}\{\tau(P)\},\text{ and } R(n, n')=\min_{P\in P_{n,n'}}\{T(P)\}.$$

In the remainder of this section we present bounds on these functions; these bounds are best possible in some cases.

2.1. THE MAXMAX PROBLEM

The number of tetrahedra used in a tetrahedrization of a given set of n points is a maximum if all $\binom{n}{2}$ pairs of points are connected by edges. We will show that for every number n' of hull vertices, $4\leq n'\leq n$, there is a tetrahedrization with $\binom{n}{2}$ edges. This will prove that the trivial upper bound, $t\leq\binom{n-1}{2}-n'+2$, can be achieved for every n'. We should remark that Rothschild & Straus (1985) also contain a lower bound for this, which is weaker but is valid in higher dimensions. Our construction uses the moment curve in four dimensions and projections of a four-dimensional polytope onto three-dimensional linear subspaces. Its validity relies on a combinatorial fact about the position of points on the moment curve.

LEMMA 2.2. *Consider the moment curve $\mathcal{M}_4=\{(x, x^2, x^3, x^4)\}$. For any four points p_0, p_1, p_2, p_3 on $\mathcal{M}_4$ determined by $x_0<x_1<x_2<x_3$, the hyperplane h determined by p_0, p_1, p_2, p_3 meets $\mathcal{M}_3$ only at $\{p_0, p_1, p_2, p_3\}$. Equivalently, the points on $\mathcal{M}_4$ given by x in $(-\infty, x_0)\cup(x_1, x_2)\cup(x_3, +\infty)$ are on the opposite side of h from the points on $\mathcal{M}_4$ given by x in $(x_0, x_1)\cup(x_2, x_3)$.*

PROOF. If not, then the moment curve has five points p_0, p_1, p_2, p_3, p_4 determined by x_0, x_1, x_2, x_3, x_4 that lie in a common hyperplane. Suppose the defining equation of the hyperplane is $a+bx+cy+dz+ew=0$. Treating these as linear equations for the unknown parameters a, b, c, d, e, we have a non-zero solution if and only if the matrix of coefficients with entries $m_{i,j}=x_{i-1}^{j-1}$, $1\leq i, j\leq 5$, has determinant zero. However, this is the Vandermonde matrix, with determinant $\Pi_{j>i}(x_j-x_i)\neq 0$.

The following discussion is simplified if we introduce an above/below relation along the fourth coordinate axis. We say that a point (x_0, y_0, z_0, w_0) lies *above* a hyperplane $A+Bx+Cy+Dz+Ew=0$ if

$$E>0\quad\text{and}\quad A+Bx_0+Cy_0+Dz_0+Ew_0>0$$

(or, equivalently, if both are negative). Otherwise, the point lies *below* the hyperplane unless it lies on it. Hyperplanes with $E=0$ are called *vertical* and the above/below relation is not defined.

Note that the parameters a, b, c, d, e of the hyperplane, h, in the proof of Lemma 2.2 are such that $\prod_{1\le i\le 4}(x-x_i)=a+bx+cy+dz+ew$; therefore, $e=1$ and the above/below relation is defined. Indeed, all points defined by x in $(x_0, x_1)\cup(x_2, x_3)$ lie below h and all points in $(-\infty, x_0)\cup(x_1, x_2)\cup(x_3, +\infty)$ lie above h.

We construct the maximum tetrahedrizations from a convex polytope, $\mathcal{P}$, defined as the convex hull of n points on $\mathcal{M}_4$. Let these be $p_0, p_1, \dots, p_{n-1}$ determined by $x_0<x_1<\dots<x_{n-1}$. $\mathcal{P}$ is a so-called cyclic polytope (see for example, Bronsted (1983), chapter 2) which satisfies the following properties.

(i) The three-dimensional faces of $\mathcal{P}$ are the tetrahedra determined by $L=\{\{p_i, p_{i+1}, p_{j+1}, p_{j+2}\}: 0\le i<j\le n-3\}$ and $U=\{\{p_0, p_i, p_{i+1}, p_{n-1}\}: 1\le i\le n-3\}$. It follows from Lemma 2.2 that $\mathcal{P}$ is supported from below by any hyperplane spanning a tetrahedron in L and supported from above by any hyperplane spanning a tetrahedron in U.

(ii) Each pair $\{p_i, p_j\}$, $i\ne j$, defines an edge of $\mathcal{P}$, and every edge is an edge of a tetrahedron in L.

(iii) $p_0, p_1, \dots, p_{n-1}$ are the n vertices of $\mathcal{P}$, and every vertex is a vertex of at least one tetrahedron in L and in U.

If we project the lower boundary of $\mathcal{P}$ (the tetrahedra in L) vertically onto the xyz-space we get a tetrahedrization with $n'=n$ hull vertices and $\binom{n}{2}$ edges. It follows that the number of tetrahedra is $\binom{n-1}{2}-n+2$ which, indeed, is the cardinality of L. In order to get maximum tetrahedrizations for $n'<n$ we move a point q on a vertical line from $w=+\infty$ downwards. The line is chosen so that it intersects $\mathcal{P}$ and so that no two hyperplanes spanning three-dimensional faces of $\mathcal{P}$ intersect it in the same point. At any location of q (still above $\mathcal{P}$) we centrally project from q onto the xyz-space all tetrahedra of $\mathcal{P}$ that q cannot see.† As q moves closer to $\mathcal{P}$ it sees fewer and fewer tetrahedra until, right before it meets $\mathcal{P}$, it sees only one tetrahedron. Whenever a tetrahedron disappears from q's sight it takes with it a vertex of $\mathcal{P}$—in other words, n' decreases by 1. Thus, for every value of n', $4\le n'\le n$, we have a tetrahedrization with n vertices altogether, n' hull vertices, and $\binom{n}{2}$ edges. This implies the following result.

THEOREM 2.3. $T(n, n')=\binom{n-1}{2}-n'+2$, *for* $4\le n'\le n$.

2.2. THE MINMIN PROBLEM

The trivial lower bound $n-3=n'-3+n''\le t$ is achieved when the tetrahedrization has exactly $3n'-6+2n''$ edges. Since $3n'-6$ edges are hull edges, we must then have $2n''$ interior edges. This is possible when $n''=0$, but already when $n''=1$ we need more interior edges and thus more tetrahedra. In fact, it seems that we need at least four edges for

† There is a minor technical difficulty which comes up when the hyperplane through q parallel to the xyz-space intersects $\mathcal{P}$. In this case, the central projection does not yield a tetrahedrization in the sense defined above. To remedy this deficiency, we choose a hyperplane that separates q and $\mathcal{P}$ and centrally project the invisible tetrahedra of $\mathcal{P}$ onto this hyperplane.

each additional interior point. The resulting bound is achievable constructively, by an iterative procedure that forms the basis for the algorithm in section 4. Rothschild & Straus (1985) consider the value of $\tau(n, n')$ in higher dimensions, allowing the inclusion of degenerate point sets.

THEOREM 2.4. *$\tau(n, n') \leq n' - 3 + 3n''$, with equality when $n'' \leq 3$.*

PROOF. With $n'' = 0$, we achieve the trivial lower bound of $n - 3$ when there are no interior edges. To construct the example inductively, add the nth point by pasting a tetrahedron onto a hull facet of the example on $n - 1$ points in a way that maintains convexity. Viewing this construction in reverse, a vertex of hull degree three (an "ear") is always available for removal. Note that this construction can be performed with each vertex incident to at most six edges, by making the nth vertex adjacent to the three previous vertices.

When $n'' > 0$, we can insert interior vertices successively into the tetrahedrization described above. Similarly, the algorithm in section 4 starts with a tetrahedrization of the hull points and then absorbs the interior points. For each new vertex, we add edges to the four corners of the tetrahedron containing it, replacing that cell by four. Each additional vertex adds four edges and three tetrahedra, so $\tau(n, n') \leq n' - 3 + 3n''$.

To show equality when $n'' \leq 3$, we must force interior edges in any such tetrahedrization. In addition to the $3n' - 6$ hull edges, let the interior edges be counted by $e_0 + e_1 + e_2$, where e_i counts the edges with i interior vertices as endpoints. Then $t = n' - 3 - n'' + e_0 + e_1 + e_2$. We show next that $e_1 + e_2 \geq 4n''$ if $n'' \leq 3$. It may hold that $e_0 + e_1 + e_2 \geq 4n''$ in general, which would make this bound optimal.

Let S be the set of interior vertices. Every interior vertex must be incident to at least four edges and to at least four tetrahedra. Now let x be an interior vertex with exactly four edges to y_1, y_2, y_3 and y_4. Since choice is limited, every 4-tuple $\{x, y_i, y_j, y_k\}$, $i, j, k \in \{1, 2, 3, 4\}$, forms a tetrahedron; in particular, every triplet $\{y_i, y_j, y_k\}$ forms a facet of the tetrahedrization. So if we remove x together with its incident edges and facets, then we are left with a tetrahedrization with S decreased by one point and $e_1 + e_2$ decreased by four. By induction, the bound is thus tight if there is an interior vertex with exactly four incident edges. If there is no such vertex, then every vertex in S is incident to at least five edge. Then

$$e_1 + e_2 \geq 5n'' - \binom{n''}{2},$$

where the binomial coefficient accounts for the edges that connect interior vertices and thus are counted twice in the first term. The claim follows because $5n'' - \binom{n''}{2} \geq 4n''$ if $n'' \leq 3$.

2.3. THE MAXMIN PROBLEM

Here, we investigate the minimum number of tetrahedra needed to tetrahedrize the worst point set. For the case $n = n'$ where all points are hull points, Sleator *et al.* (1986) uses hyperbolic geometry to show there exists a point set for which every tetrahedrization has at least $2n - 10$ cells when $n > 12$. This shows that the algorithm and resulting bound in the next theorem are optimal when all points are hull points. It would be nice to extend their proof to consider interior points, or to obtain a combinatorial proof of the lower bound for the case when all points are hull points.

THEOREM 2.5. $\rho(n, n') \leq 3n - n' - 10$ for $n' > 12$, *with* 10 *replaced by* 9, 8, 7, *respectively, for* $n' \in \{7, 8, 9, 10, 11, 12\}, \{5, 6\}, \{4\}$.

PROOF. Again we use the insertion algorithm to obtain a linear-size tetrahedrization for any point set. Begin by considering the hull points. Let Δ be the maximum vertex degree in the graph of the boundary of the convex hull, and let v be a vertex attaining that. We call Δ the *hull degree of* v. Add edges from v to the $n' - 1 - \Delta$ vertices not adjacent to it on the boundary of the convex hull and add facets from v to the edges that connect these vertices. This tetrahedrizes the interior with $2n' - 4 - \Delta$ cells. Now consider each interior point successively, and add edges to the four corners of the cell containing it, replacing that cell by four cells. The resulting number of cells is $2n' - 4 - \Delta + 3n'' = 3n - n' - 4 - \Delta$. The graph of the boundary of the convex hull is an arbitrary maximal planar graph, which may have Δ as small as $\lceil 6 - 12/n \rceil$.

If $\Delta = n' - 1$, we have constructed a tetrahedrization of size $n' - 3 + 3n''$. This suggests a more refined problem. Let

$$\rho(n, n', \Delta) = \max_{P \in \mathbf{P}^{\Delta}_{n,n'}} \{\tau(P)\},$$

where

$$\mathbf{P}^{\Delta}_{n,n'} = \{P \in \mathbf{P}_{n,n'}\colon \text{largest hull degree is equal to } \Delta\}.$$

We have $\rho(n, n', \Delta) \leq 2n' - 4 - \Delta + 3n''$ and conjecture equality. This is the most important of the combinatorial problems discussed in this section, since the truth of this conjecture would demonstrate that the algorithm of section 4 is optimal in terms of the number of tetrahedra it constructs.

2.4. THE MINMAX PROBLEM

In this section, we study the smallest number R such that every set of n points in general position, with n' hull points, has a tetrahedrization of size at least R. Tetrahedrizations with many cells are among the least useful, from many practical viewpoints. Consequently, we consider this extremal problem the least significant of the four considered. The best lower bound we have is the trivial one applicable to all tetrahedrizations; that is, this extremal problem is also the one where we have the weakest results. More precisely, there are point sets for which the largest known tetrahedrization is linear. However, we have not been able to bring the upper bound below quadratic; we have been able to forbid only a constant fraction of the edges. Indeed, we show an upper bound on $R(n, n')$ which is approximately 14/15 times $T(n, n')$.

THEOREM 2.6. $R(n, n') \geq 3n'' + f(n')$, *where* $f(n') = 4n' - 25$ *if* $13 \leq n'$, $f(n') = 3n' - 13$ *if* $7 \leq n' \leq 12$, $f(n') = 2n' - 7$ *if* $5 \leq n' \leq 6$, *and* $f(n') = n' - 3$ *if* $n' = 4$. *For* $n' = n$,

$$R(n, n') \leq \binom{n}{2} - \frac{n^2}{30} + O(n).$$

PROOF. We give a construction of a tetrahedrization that shows the lower bound. Consider first the case $n''=0$. Take the vertex, v, with the maximum degree, d, and remove it from the point set. Recompute the convex hull and fill the space between v and the new hull with $d-2$ tetrahedra. These tetrahedra are spanned by v and the triangles of the new convex hull that v can see. Iterate on this process until only three points are left. To obtain the lower bound stated in the theorem just note that $d \geq 6$ as long as there are at least 13 vertices, $d \geq 5$ as long as there are at least seven vertices, $d \geq 4$ if the number of vertices is five or six, and $d=3$ if there are four vertices. To generalize this result to arbitrary n'' note that an interior vertex gives rise to two additional triangles on the convex hull when it appears on the hull for the first time. Thus, it also gives rise to two additional tetrahedra. When we remove this point as a vertex of the hull its degree is at least three which gives rise to another tetrahedron.

Proving upper bounds on $R(n, n')$ is equivalent to finding point sets for which every tetrahedrization has many missing edges. The smallest point set for which not all edges can appear in a tetrahedrization is the set of six points whose hull edges form the graph of the octahedron. In addition to the 12 hull edges, there are three skew diagonals joining opposite vertices. If all 15 edges are used, Lemma 2.1 yields six tetrahedra, and then Euler's formula says there are 16 facets. However, each diagonal belongs to at least three facets, and since the diagonals share no endpoints these facets are all distinct. Together with the eight hull facets, we get at least 17 facets and a contradiction.

It is interesting to note that all three diagonals of an octahedron can be used if additional points are used as vertices. However, at least one such vertex must be inside the octahedron. The proof of this claim is left as an exercise to the reader. We will use this condition to our advantage.

Now suppose $n=n'$ and the points are clustered in six groups near the six vertices of an octahedron. As the points in each group approach that vertex of the octahedron, the only cells that retain positive volume are those using points from four clusters with at least one cluster from each diagonally opposite pair of vertices. If an edge is used between each pair of opposite clusters, then we obtain a tetrahedrization of the octahedron using all three diagonals, which was forbidden above. Hence there are no edges joining some pair of opposite clusters, which means there are at least $\lfloor n/6 \rfloor^2$ edges missing. We distribute the points within each cluster recursively by the same argument and so that all points are convex hull vertices. The forbidden edges are counted by a geometric series summing to $n^2/30+O(n)$.

3. A Geometric Partitioning Problem

In this section, we examine a partitioning problem for finite point sets inside a simplex in d dimensions. The three-dimensional case will be applied later when we discuss a time-optimal algorithm that tetrahedrizes a finite set of points in three dimensions. The objective is to choose a point that will partition the other points in a balanced way. The problem is to guarantee that there is a point that does a good job of balancing. More precisely:

Let P be a set of n points in a d-dimensional simplex $\mathcal{S}$ with vertex set $V=\{v_i : 1 \leq i \leq d+1\}$. We assume that no four points of $P \cup V$ are co-planar. For any $p \in P$, let $\mathcal{S}_i(p)$ be the simplex with vertices $\{p\} \cup (V-\{v_i\})$; note that $\bigcup_{i=1}^{d+1} \mathcal{S}_i(p)=\mathcal{S}$. We define the *imbalance of* a point $p \in P$ to be $\beta(p)=\max_{1 \leq i \leq d+1}\{|P \cap \mathcal{S}_i(p)|-1\}$. The problem is to determine the smallest imbalance that can be guaranteed; that is, the value of $\beta_d(n) = \max_P \min_{p \in P}\{\beta(p)\}$, where P runs over all n-point sets in a d-dimensional simplex $\mathcal{S}$.

The main result of this section determines the exact value of $\beta_d(n)$.

THEOREM 3.1. $\beta_d(n) = \lfloor d \cdot n/(d+1) \rfloor$.

PROOF. Suppose $n \equiv k \bmod (d+1)$. To prove $\beta_d(n) \geq \lfloor d \cdot n/(d+1) \rfloor$, let c be an interior point of $\mathcal{S}$ and place the points of P arbitrarily close to the line segments cv_i. For $1 \leq i \leq k$, place $\lceil n/(d+1) \rceil$ points close to cv_i; with $i > k$ associate $\lfloor n/(d+1) \rfloor$ points. For any point p associated with any line segment cv_i, the points associated with all other line segments all lie in the simplex $\mathcal{S}_i(p)$. We can avoid any additional imbalance by letting p be the innermost point associated with cv_1 (see Figure 1(a)). Hence for this P we have $\min_{p \in P}\{\beta(p)\} = n - \lceil n/(d+1) \rceil = \lfloor d \cdot n/(d+1) \rfloor$.

To prove the upper bound, consider an arbitrary n-point set P in $\mathcal{S}$; we find $p \in P$ with $\beta(p) \leq n - \lceil n/(d+1) \rceil$. Let $\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_{d+1}$ denote the $d+1$ facets of $\mathcal{S}$, with the vertices of $\mathcal{F}_i$ being $V - \{v_i\}$. From each facet $\mathcal{F}_i$ choose a $(d-2)$-face of $\mathcal{S}$; call it $\mathcal{F}'_i$. Among the hyperplanes through the interior of $\mathcal{S}$ that intersect $\mathcal{F}_i$ in precisely $\mathcal{F}'_i$, let h_i be the one through a point p_i of P that has exactly $\lceil n/(d+1) \rceil - 1$ points of P on the opposite side of it from $\mathcal{F}_i$. Call this set of points P_i (see Figure 1(b)). Define $\bar{P} = P - \bigcup_{1 \leq i \leq d+1} P_i$. By construction,

$$|\bar{P}| \geq n - (d+1)\left(\left\lceil \frac{n}{d+1} \right\rceil - 1\right) > 0.$$

Thus, $\bar{P}$ is non-empty. For any $p \in \bar{P}$, $\mathcal{S}_i(p)$ contains no point of $P_i \cup \{p_i\}$. Consequently, for every i, $\mathcal{S}_i(p)$ contains at most

$$n - \left\lceil \frac{n}{d+1} \right\rceil = \left\lfloor \frac{d \cdot n}{d+1} \right\rfloor$$

points of P.

Notice that the argument that proves the upper bound on $\beta_d(n)$ is constructive. In fact, it suggests an algorithm that can be implemented to run in time $O(n)$:

for $i := 1$ **to** $d+1$ **do**

Choose $\mathcal{F}'_i$ from facet $\mathcal{F}_i$ and construct hyperplane h_i as defined above using a linear time median algorithm (see Aho *et al.*, 1974). Mark the points of P that lie on the opposite side of h_i from $\mathcal{F}_i$.

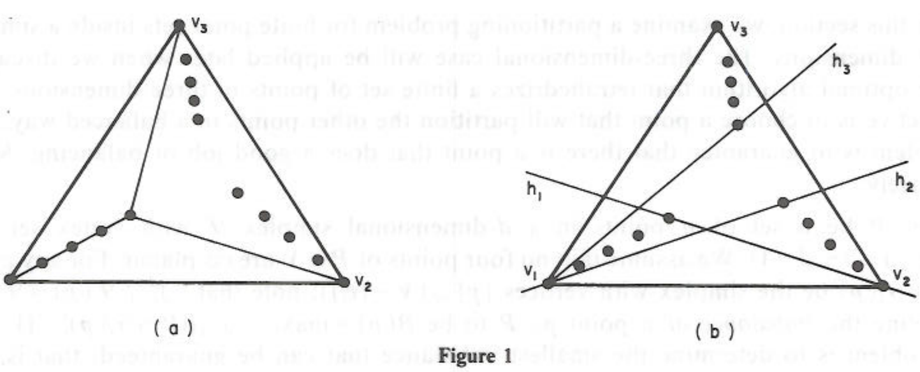

Figure 1

endfor;

The unmarked points of P are exactly those of set $\bar{P}$. Partition $\mathcal{S}$ using an arbitrary point $p \in \bar{P}$.

We conclude the following result.

THEOREM 3.2. *Let $\mathcal{S}$ be a simplex with vertex set V, and let P be a set of n points in the interior of $\mathcal{S}$ such that no four points of $P \cup V$ are co-planar. Then there is a linear-time algorithm that picks a point p of P such that each simplex spanned by p and a facet of $\mathcal{S}$ contains at most $\beta_d(n)$ points of P.*

Intuitively, Theorem 3.2 states that there is a time-optimal algorithm that constructs a partition that is at least as good as the best partition for the worst point set.

4. Tetrahedrizing by Stepwise Refinement

This section gives an algorithm that constructs a tetrahedrization of a set P of n points in E^3 in $O(n \log n)$ time. The size of the constructed tetrahedrization is

$$2n' - 4 - \Delta + 3n'',$$

where n'' of the points are interior and Δ is the largest hull degree.

The algorithm has two phases. First, it constructs the convex hull of P and a tetrahedrization of the hull points of P. Second, it iteratively inserts the interior points by subdividing the tetrahedron that contains such a point into four tetrahedra. Below, we give a formal description of this process. We assume that no four points of P are co-planar.

ALGORITHM 4.1. (Tetrahedrizing by stepwise refinement.)

Initial step. Construct the convex hull of P, pick a vertex v with largest degree Δ, and initialize the tetrahedrization to the set of tetrahedra spanned by v and the hull facets that do not contain v. For each tetrahedron $\mathcal{T}$ constructed, determine $P \cup int(\mathcal{T})$, the set of points in P contained in the interior of $\mathcal{T}$, and push $\mathcal{T}$ onto a stack.

Iteration. Refine the tetrahedrization as follows:

while there is a tetrahedron $\mathcal{T}$ on the stack **do**

Remove $\mathcal{T}$ from the stack.

if $m = |P \cap int(\mathcal{T})| \geq 1$ **then**

Pick a point p in $P \cap int(\mathcal{T})$ with imbalance at most $\beta_3(m)$ and partition $\mathcal{T}$ into four tetrahedra spanned by p and the facets of $\mathcal{T}$. For each new tetrahedron $\mathcal{T}'$, determine $P \cap int(\mathcal{T}')$ and push it onto the stack.

endif

endwhile.

To implement the initial step of Algorithm 4.1, we need an algorithm that constructs the convex hull of P and an algorithm that determines the location of each interior point in the initial tetrahedrization. For the first part, we use the algorithm of Preparata & Hong (1977) which takes $O(n \log n)$ time to construct the convex hull of P.

Due to the special structure of the initial tetrahedrization, we can reduce the second part of this step to a batched version of the planar point location search problem. Let v be the maximum degree vertex of the convex hull $\mathcal{P}$ of P chosen by Algorithm 4.1, and let h be a plane through v that is tangent to $\mathcal{P}$. Let h' be a plane parallel to h, with $\mathcal{P}$

between them. Project every facet of $\mathcal{P}$ from point v onto plane h'. This yields a triangulation in h' with the property that an interior point p of P belongs to a tetrahedron spanned by v and facet $\mathcal{F}$ if and only if the central projection of p onto h' belongs to the projection of facet $\mathcal{F}$ onto h'. There are several algorithms that solve the batched point location problem in $O(n \log n)$ time (see Preparata, 1979; Lee & Yang, 1979; Edelsbrunner *et al.*, 1986). Hence the initial step of Algorithm 4.1 can be implemented in $O(n \log n)$ time.

The only non-trivial part of the iteration step of Algorithm 4.1 is the determination of a point within a given tetrahedron $\mathcal{T}$ with imbalance at most $\beta_3(m)$, where $m = |P \cap int(\mathcal{T})|$. By Theorem 3.2, such a point can be found in $O(m)$ time.

For a particular input, let T_I be the total amount of time required by the iteration step. Let n' be the number of hull points of P, and notice that the tetrahedrization constructed in the initial step of Algorithm 4.1 has $t' = 2n' - 4 - \Delta$ cells. We have

$$T_I = \sum_{i=1}^{t'} T(n''_i),$$

where n''_i is the number of points in the ith cell of the initial tetrahedrization and $T(n''_i)$ is the amount of time required to tetrahedrize it. Using recursive partitioning by balanced points,

$$T(n''_i) = O(n''_i) + T(n''_{i,1}) + T(n''_{i,2}) + T(n''_{i,3}) + T(n''_{i,4}),$$

with $n''_{i,1} + n''_{i,2} + n''_{i,3} + n''_{i,4} = n''_i - 1$ and $\max\{n''_{i,1}, n''_{i,2}, n''_{i,3}, n''_{i,4}\} \le \beta_3(n''_i)$. Therefore, $T(n''_i) = O(n''_i \log n''_i)$ and $T_I = O(n \log n)$, which implies that Algorithm 4.1 runs in $O(n \log n)$ time.

Next, we analyse the number of tetrahedra constructed by Algorithm 4.1. The tetrahedrization constructed by the initial step consists of $2n' - 4 - \Delta$ tetrahedra, where Δ is the largest hull degree. For each interior point, we get three additional tetrahedra. This implies the main result of this section.

THEOREM 4.1. *Let P be a set of n points in three dimensions, no four co-planar, let n' be the number of hull points of P, and let n'' be the number of interior points of P. Algorithm* 4.1 *constructs a tetrahedrization of P in $O(n \log n)$ time that consists of $2n' - 4 - \Delta + 3n''$ tetrahedra.*

There are several open questions raised by the investigations described in this section. One is whether or not the constructed tetrahedrization is optimal in the worst case, that is, whether or not there exists a point set that cannot be tetrahedrized using fewer tetrahedra, for every pair of values $n' + n'' = n$. Partial results on this problem can be found in section 2.3.

There are a few questions related to the determination of a point inside a tetrahedron that has reasonably low imbalance. The algorithm suggested in section 3 determines a point that is worst-case optimal, but for a given point set there may be points with considerably lower imbalance. How expensive is it to identify a point with minimum imbalance? Also, it is not hard to see that our example of a point set inside a tetrahedron which realizes the largest minimum imbalance (see section 3) cannot have another worst-case configuration in any of the four subtetrahedra. This raises the question of how imbalanced the best partition must be in the amortized sense. To formalize this question, we define the *depth* $\delta(\mathcal{T})$ *of* each tetrahedron $\mathcal{T}$ created in the iterative partitioning process. If $\mathcal{T}$ is created during the initial step of Algorithm 4.1, put $\delta(\mathcal{T})=0$. Otherwise, $\mathcal{T}$ is obtained by partitioning an earlier tetrahedron $\mathcal{T}'$, in which case set $\delta(\mathcal{T})=\delta(\mathcal{T}')+1$. Theorem 3.1 implies that the depth of any tetrahedron is bounded from above by $\log_{4/3} n$. However, the largest necessary depth we have found in tetrahedrizing an n-point set is $\log_2 n+O(1)$: take four hull points and let the other points lie on a circular arc connecting two hull points.

Finally, there is the problem of generalizing Algorithm 4.1 to four and higher dimensions. Unfortunately, the initial step becomes by far the most expensive part of the algorithm. The worst-case complexity of any algorithm that constructs a simplicial dissection of the convex hull of a set of n points in $d \geq 2$ dimensions cannot be less than $O(n^{[d/2]})$, since there are examples such that the convex hull itself consists of $O(n^{[d/2]})$ facets (see Bronsted, 1983). More about this problem can be found in Avis & ElGindy (1987).

5. Other Tetrahedrization Methods

As noted earlier, the Delaunay triangulation of a planar point set has the interesting property of local equiangularity, that is, no diagonal of a convex quadrilateral comprising two adjacent triangles can be flipped to achieve an increase of the minimum of the six angles of the triangles. This is viewed as an indication of how well-proportioned it is and, although it is not clear how this notion generalizes to three dimensions, it motivates algorithms for the Delaunay tetrahedrization.

Reported in the literature (Cavendish *et al.*, 1985) is an approach to Delaunay tetrahedrization, based on iterative insertion, which may run in $\Omega(n^3)$ time for a worst-case point set of size n. We propose the following technique which is folklore but to the best of our knowledge not described anywhere in the literature. We consider the three-dimensional space E^3 containing P (with coordinates x, y and z) as the hyperplane $w=0$ in the four-dimensional space E^4 with coordinates x, y, z and w. Next we project the points of P in the w-direction to the rotation paraboloid of equation $w=x^2+y^2+z^2$. Let $\phi(P)$ be the resulting set of points. As discussed in Edelsbrunner & Seidel (1986) and Preparata & Shamos (1985) this mapping (the composition of an inversive and a projective transformation) identifies the Delaunay tetrahedrization in E^3 with an appropriate portion of the convex hull of $\phi(P)$ in E^4. Therefore, the construction can be carried out by resorting to a four-dimensional convex hull algorithm; the most efficient one is due to Seidel (1981) and runs in time $O(n^2)$. We summarize:

THEOREM 5.1. *The Delaunay tetrahedrization of a set of n points in E^3 can be constructed in time $O(n^2)$.*

We now discuss an alternative algorithm to that of section 4 to produce a linear-size tetrahedrization. The shortcoming of this algorithm is its substantially higher running time, but the shapes of the cells produced are likely to be better proportioned.

Let (P', P'') be a partition of P so that P' is the set of hull points of P. By the assumption of non-degeneracy, the edges on the boundary of the convex hull of P form a maximal planar graph $G(P)$ with $|P'|$ vertices and $3|P'|$-6 edges, and therefore contains a vertex of hull degree at most five.

An iteration of the algorithm selects a minimum degree vertex v of $G(P)$, and constructs the convex hull of $P-\{v\}$, until the current point set consists of exactly four points. This action has motivated the name of "Greedy Peeling".

By means of a list containing the vertices of degree at most five, selection of v is trivial. The bulk of the work is the construction of the new convex hull. The required action is a local "gift-wrapping", which constructs one facet at a time. The construction of a facet can be viewed as a plane pivoting around an existing edge until it first meets a point of $P-\{v\}$. In the dual space, pivoting is interpreted as tracing an edge of the skeleton of an arrangement of planes, which can be done in time $O(n^{1/2}\log n)$ and $O(n)$ memory space by means of the algorithm reported in Edelsbrunner (1986).

To evaluate the size of the resulting tetrahedrization, we consider the set difference $\mathscr{C}(p)$ between the convex hull of P and the convex hull of $P-\{p\}$. Let ν denote the number of vertices of the (non-convex) polytope $\mathscr{C}(p)$ that belong to P''. Out of ν vertices of $\mathscr{C}(p)$ at most six are vertices of the convex hull of P (this includes the vertex p). It is readily verified that $\mathscr{C}(p)$ can be tetrahedrized into at most $3+2\nu$ cells: of these, we "charge" three to vertex v, and two to each of the ν points which have emerged as new hull vertices. Since this extra charge occurs only once for points of P'', the total number of tetrahedra generated by the algorithm is bounded above by $3n'+5n''-11\le 5n-19$. Therefore the greedy peeling algorithm produces a linear-size tetrahedrization of a set of n points in E^3 in time $O(n^{3/2}\log n)$.

6. Discussion and Open Problems

For convenience, we assumed in our discussion of tetrahedrizations that no four points are co-planar. In practical applications this will not always be the case. To render our results useful for real life computations, we suggest that the input points be conceptually perturbed. If the perturbation is defined for every $\varepsilon>0$ and the perturbed set approaches the input set when ε goes towards 0, then all computations can be based on the assumption that ε is sufficiently small and no suitable value of ε needs to be computed. More details about such a method and its efficient implementation can be found in Edelsbrunner (1986). The disadvantage of the perturbation method is that the algorithm constructs tetrahedra whose vertices are co-planar or collinear. Depending on the application, we may or may not want to remove such tetrahedra in a final phase. If we remove the degenerate tetrahedra, we will be left with vertices that lie on edges or facets of the tetrahedrization. Thus, the obtained cell complex will not necessarily be face-to-face.

Finally, we mention several interesting questions about tetrahedrizing point sets.

(i) What is the minimum and maximum difference or ratio between $\tau(P)$ and $T(P)$? In particular, the point sets achieving $\tau(n, n')$ and $T(n, n')$ or $\rho(n, n')$ and $R(n, n')$ are different.

(ii) What are the minimum and maximum sizes of Delaunay tetrahedrizations for point sets in $P_{n,n'}$? Seidel (1982) gives the maximum for $n'=4$. In this case, the numbers are the same as for general tetrahedrizations (see Theorem 2.3).

(iii) For a given point set P do there exist tetrahedrizations of all sizes between $\tau(P)$ and $T(P)$? This would follow if one could show that every tetrahedrization of P can be transformed into any other tetrahedrization by a sequence of local changes, where a local change replaces an appropriate facet by the edge connecting the respective forth vertices of the two tetrahedra, or vice versa.

(iv) Can the region between two non-overlapping convex polytopes always be tetrahedrized with a linear number of tetrahedra?

(v) Is there an algorithm for the Delaunay tetrahedrization for point sets in $P_{n,n'}$ that runs in time $O(n(\log n)^{\alpha}+k)$, for some $\alpha>0$, where k is the size of the tetrahedrization?

The motivation for the last three questions is that of finding reasonably efficient tetrahedrizations using tetrahedra that are well-shaped. Our fast algorithm for finding a linear tetrahedrization tends to use elongated tetrahedra.

REMARK. After finishing the research on the presented subject, the authors learned that the $O(n \log n)$ time construction of a tetrahedrization in three dimensions has been discovered independently by Avis & ElGindy (1987). Avis & ElGindy also offer an elaborate discussion of degenerate point sets that possibly contain co-planar and collinear points. Their method to cope with degenerate point sets is rather different from the one suggested in section 6.

The authors wish to thank Emo Welzl for substantially simplifying part of the proof of Theorem 2.4.

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