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A Hyperplane Incidence Problem with Applications to Counting Distances

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ABSTRACT. This paper proves an $O(m^{2/3}n^{2/3}+m+n)$ upper bound on the number of incidences between m points and n hyperplanes in four dimensions, assuming all points lie on one side of each hyperplane and the points and hyperplanes satisfy certain natural general position conditions. This result has applications to various three-dimensional combinatorial distance problems. For example, it implies the same upper bound for the number of bichromatic minimum distance pairs in a set of m blue and n red points in three-dimensional space. This improves the best previous bound for this problem.

1. Introduction

Combinatorial distance problems for finite point sets in Euclidean spaces are classical topics in discrete geometry. Most popular is probably the question of how often the unit distance can occur in a set of m points in two or three dimensions; this question was originally studied by Erdös [**10, 11**]. The current best upper bound in the plane is $O(m^{4/3})$ (see [**15**] and also [**4**]), and in three dimensions Clarkson et al. [**4**] proved $O(m^{3/2}\beta(m))$, where $\beta(m)$ is an extremely slowly growing function related to the inverse of Ackermann's function. No matching lower bounds are known.

This paper considers three distance problems in three dimensions and improves the best previous upper bound in each case. The results are as follows. The number of bichromatic minimum distance pairs in a set of m blue and

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n red points is $O(m^{2/3}n^{2/3}+m+n)$. The same bound holds for the number of bichromatic nearest neighbor pairs if no three points are collinear. For a (monochromatic) set of m points, the number of furthest neighbor pairs is $O(m^{4/3})$ if no three points are collinear. All three bounds are corollaries of the following more general result on the number of incidences between m points and n hyperplanes in four dimensions. If no three points are collinear, no three hyperplanes intersect in a common 2-flat, and each hyperplane bounds a closed half-space that contains all points, then the number of incidences is $O(m^{2/3}n^{2/3}+m+n)$. Besides the application of this result to three-dimensional distance problems we also consider an application to three-dimensional Delaunay triangulations.

This paper is organized as follows. §2 proves the upper bound on the hyperplane incidence problem, §3 explains how this problem relates to counting incidences between points and spheres in three dimensions, §4 discusses the combinatorial distance problems, and §5 concludes the paper.

2. The hyperplane incidence problem

This section proves the main result of this paper, an upper bound for the following combinatorial geometry problem.

PROBLEM SPECIFICATION. Let P be a set of m points and H a set of n hyperplanes in four-dimensional Euclidean space satisfying the following conditions.

(H.i) No three points of P are collinear.
(H.ii) No three hyperplanes of H intersect in a common 2-flat.
(H.iii) Each hyperplane in H bounds a closed half-space that contains P.

What is the maximum number of incidences between the points and the hyperplanes, in terms of m and n, where the maximum is taken over all sets P of size m and H of size n satisfying the three conditions? We define $I(m, n)$ to be equal to this number.

We prove an upper bound on $I(m, n)$ by adapting the methods of Clarkson et al. [4]. This is done in two steps. First, bounds are obtained which are tight for the cases when m is much smaller than n $(m < \sqrt{n})$ and when n is much smaller than m $(n^2 < m)$; using the terminology of [4] we call these bounds Canham thresholds. Second, these bounds together with the methods of [4] are used to establish an upper bound for the remaining case $(\sqrt{n} \leq m \leq n^2)$. To prove the Canham thresholds (Lemma 2.1) we make use of the fact that the problem is self-dual, that is, if we apply a dual transform we arrive at the same problem, only with the roles of m and n interchanged. Indeed, choose the origin in the interior of the intersection of half-spaces that contains all points of P. For a point p different from the origin let

$$p^* = \{x \mid \langle x, p\rangle = 1\}$$

be the *polar hyperplane*, and for a hyperplane h avoiding the origin define the *polar point* h^* so that $h = (h^*)^*$. Using straightforward algebraic manipulations it can easily be verified that $p \in h$ iff $h^* \in p^*$. Furthermore, three points are collinear iff their polar hyperplanes intersect in a common 2-flat. In other words, if we map all points to their polar hyperplanes and all hyperplanes to their polar points we get a point/hyperplane incidence problem satisfying conditions (H.i) and (H.ii). In addition to the incidence preserving property the polar transform preserves sidedness relative to the origin, which implies that also condition (H.iii) is satisfied after polarization. Hence, $I(m, n) = I(n, m)$.

The Canham thresholds. The intersection of half-spaces defined by the hyperplanes is a closed convex polyhedron $\mathscr{C}$ with at most n facets. The upper bound theorem (see **[3]** or **[8]**) implies that $\mathscr{C}$ has at most $O(n^2)$ vertices, edges, and ridges.[1] Let the *degree* of a face (vertex, edge, ridge, or facet) be the number of hyperplanes in H that contain it. Clearly, the degree of any facet is 1, by assumption (H.ii) the degree of any ridge is 2, and the degree of any edge or vertex can be arbitrary. Still, the sum of degrees, over all faces, is at most $O(n^2)$ by the following argument. Take each hyperplane that contains a vertex or edge of $\mathscr{C}$ and push it inwards (towards the origin) by a tiny amount so that each such hyperplane now supports a facet. Call the new polytope $\mathscr{C}'$. Next we slightly perturb the hyperplanes, on a much smaller scale than before, in order to transform $\mathscr{C}'$ into a simple polyhedron $\mathscr{C}''$, again without decreasing the sum of face degrees. Now every edge has degree 3 and every vertex has degree 4 which implies the claim by applying the upper bound theorem to $\mathscr{C}''$. After these introductory remarks we are ready to prove the Canham thresholds.

LEMMA 2.1. *$I(m, n) = O(m\sqrt{n} + n)$ and $I(m, n) = O(n\sqrt{m} + m)$.*

PROOF. Because of the self-dual property of the incidence problem we can restrict ourselves to proving $I(m, n) = O(n\sqrt{m}+m)$. Let P be the set of m points, H the set of n hyperplanes, and $\mathscr{C}$ the polyhedron defined above which contains P. If a point $p \in P$ lies in the interior of $\mathscr{C}$ it contributes 0 to the number of incidences, if it lies in the relative interior of a facet it contributes 1, and if it lies in the relative interior of a ridge it contributes 2. The total number of incidences involving such point is therefore $O(m)$. Each edge of $\mathscr{C}$ contains at most two points of P, by condition (H.i), and each vertex contains at most one point of P. By the claim established earlier, the total number of incidences involving points on edges and vertices of $\mathscr{C}$ is therefore $O(n^2)$ which gives a combined bound of $O(m + n^2)$. To get $I(m, n) = O(n\sqrt{m}+m)$ we simply partition H into $\lceil n/\sqrt{m} \rceil$ subsets of size at most $\sqrt{m}$ each. For each subset we get only $O(m)$ incidences with points of P and therefore we get at most $O(n\sqrt{m} + m)$ incidences altogether. □

[1] In general, a *ridge* is a $(d-2)$-face of a d-polyhedron. Because $\mathscr{C}$ is a 4-polyhedron a ridge of $\mathscr{C}$ is a 2-face.

REMARKS. (1) Conditions (H.i) and (H.ii) can be relaxed to allow up to some constant number of points that are collinear and hyperplanes that meet in a common 2-flat without sacrificing the asymptotic bounds in Lemma 2.1. However, if we drop (H.i) or (H.ii) we can find examples with $\Omega(mn)$ incidences. The problem that results when we drop (H.iii) but not (H.i) and (H.ii) is more difficult. The best lower bound known to the authors is $\Omega(m^{4/3}\log\log m)$ for the case $m = n$ and is based on an example of Erdös [11] (see also [4]).

(2) Conditions (H.i) and (H.ii) imply that no three hyperplanes can all be incident to three common points. Using a standard extremal graph lemma (see [2] or [4]) this can be used to prove $O(mn^{2/3}+n)$ and $O(nm^{2/3}+m)$ as upper bounds on $I(m, n)$. These bounds are significantly weaker than the bounds of Lemma 2.1, which illustrates the importance of condition (H.iii) and the use of the upper bound theorem.

The main result. We introduce some notation. For a hyperplane $h \in H$ let h^+ be the closed half-space bounded by h that contains the origin. Recall that we assume that the origin lies in the interior of $\mathscr{C}$ which implies that $\mathscr{C} = \bigcap_{h\in H} h^+$. We now present a sequence of arguments that add up to a proof of the main result of this section.

First, choose a random sample $R \subseteq H$ of size r. Form the convex polyhedron $\mathscr{R} = \bigcap_{h\in R} h^+$ and note that $\mathscr{R}$ contains all points of P and also the origin. It will be convenient to assume that $\mathscr{R}$ is actually a polytope (that is, it is bounded) which can be achieved by intersecting it with a sufficiently large tetrahedron.

Second, we triangulate $\mathscr{R}$ as follows. Choose a directed line so that no hyperplane normal to this line contains two vertices of $\mathscr{R}$ and call the direction defined by this line *vertical*. It thus makes sense to talk about a point being *higher* than another point. A ridge (that is, 2-face) of $\mathscr{R}$ is triangulated by connecting its highest vertex to all other vertices. Each triangle is thus bounded by two edges incident to the highest vertex of the ridge and a third edge which is an original edge of the ridge. Similarly, a facet (that is, 3-face) of $\mathscr{R}$ is triangulated by connecting its highest vertex to all vertices, edges, and triangles in the (triangulated) boundary of the facet. Thus, each tetrahedron is incident to the highest vertex and is bounded by a triangle not incident to this vertex. Finally, the interior of $\mathscr{R}$ is triangulated by connecting the origin to all vertices, edges, triangles, and tetrahedra in the (triangulated) boundary of $\mathscr{R}$. The number of 4-simplices generated is equal to the number of tetrahedra in the boundary of $\mathscr{R}$. In turn, the number of tetrahedra is at most twice the number of triangles because any ridge is incident to only two facets. Finally, the number of triangles used to triangulate the ridges is bounded from above by the number of edge/ridge incidences. The number of ridges incident to a single edge is at most the degree of the edge, and we noted earlier that the sum of edge degrees is at most quadratic in the number of hyperplanes. Thus, if k is the number of 4-simplices in the triangulation then $k = O(r^2)$.

For the application of a probabilistic counting result due to Clarkson and Shor [5] in the fifth step below, it is important that each 4-simplex is uniquely determined by some constant number of hyperplanes. Indeed, a triangle lies in a plane (the intersection of two hyperplanes) and each vertex is defined by two lines (the intersection of this plane with two other hyperplanes). At least two of the three vertices share a line because at least one of the edges of the triangle is chosen to lie on a line. Thus, we need at most seven hyperplanes to define a triangle. A tetrahedron connects a triangle to a vertex (the intersection of four hyperplanes, one of which contains the triangle) and is thus defined by at most ten hyperplanes. A 4-simplex just connects a tetrahedron with the origin and is therefore determined by at most ten hyperplanes and a point that is fixed independent of $\mathscr{R}$.

Another necessary property for the application of the probabilistic counting result is that a 4-simplex defined by the origin and at most ten hyperplanes in the way described above is in the triangulation *if and only if* it does not intersect any other hyperplane. Unfortunately, this is not strictly true because the hyperplanes are not in general position which allows for the possibility that different sets of at most ten hyperplanes define the same 4-simplex. However, we can simulate an arbitrarily small perturbation of the hyperplanes to get them into general position and define the triangulation with respect to this perturbation. The perturbation is used merely for the purpose of assigning proper sets of hyperplanes to the 4-simplices of the triangulation—it is ignored as far as point/hyperplane incidences are concerned. Before we go on with the proof we remark that the origin is an arbitrary point in the interior of $\mathscr{C}$. We may therefore assume that each point of P that does not lie on the boundary of $\mathscr{R}$ lies in the interior of a 4-simplex of the triangulation.

The third step of the proof bounds the number of incidences involving points in the interior of $\mathscr{R}$. For $1 \leq i \leq k$, let σ_i be the interior of the ith 4-simplex, set $m_i = |P \cap \sigma_i|$, and let n_i be the number of hyperplanes in H that have nonempty intersection with σ_i. Using the first bound in Lemma 2.1 we thus get $\sum_{i=1}^{k} O(m_i\sqrt{n_i} + n_i)$ as an upper bound for the number of incidences that happen in the interior of $\mathscr{R}$.

Fourth, we bound the number of incidences involving points on the boundary of $\mathscr{R}$. For a hyperplane $h \in R$ define $P_h = P \cap h$ and $m_h = |P_h|$. The convex hull of P_h is a convex polytope of dimension 3 or less. Because of its low dimensionality, it has at most $O(m_h)$ edges and 2-faces. Since no three points in P_h are collinear (condition (H.i)) and no three hyperplanes in H meet in a common 2-flat (condition (H.ii)), the total number of incidences involving points of P_h is $O(m_h + n)$. If we sum this bound over all r hyperplanes $h \in R$ we get $O(I(m, r) + nr)$ since a point p belongs to as many sets P_h as there are incident hyperplanes h in R. We have $I(m, r) = O(r\sqrt{m} + m)$ using the second bound in Lemma 2.1.

In the fifth and final step we make use of the probabilistic counting result of Clarkson and Shor [**5**]. It implies that the expected value of $\sum_{i=1}^{k} O(m_i\sqrt{n_i}+n_i)$ is $O(m\sqrt{n/r}+kn/r)$. Note that $kn/r = O(nr)$ and that the above sum is an upper bound on the number of incidences that happen in the interior of $\mathscr{R}$. Thus, there exists a subset $R \subseteq H$ of size r so that the number of incidences in the interior of $\mathscr{R}$ is at most $O(m\sqrt{n/r} + nr)$. Adding to this bound the incidences on the boundary of $\mathscr{R}$ we get

$$I(m, n) = O(m\sqrt{n/r} + nr + r\sqrt{m} + m)$$

as an upper bound on the total number of incidences. If we choose r equal to $\lceil m^{2/3}/n^{1/3} \rceil$ this bound becomes $O(m^{2/3}n^{2/3} + (m^{7/6}/n^{1/3}) + m)$. This choice of r is meaningful if $\sqrt{n} \le m \le n^2$ in which case $m^{7/6}/n^{1/3} \le m$. For the remaining cases, $m < \sqrt{n}$ and $n^2 < m$, Lemma 2.1 shows that the number of incidences is at most $O(m + n)$. This concludes the proof of the main result of this section which we now state.

THEOREM 2.2. *The maximum number of incidences between* m *points and* n *hyperplanes in four dimensions satisfying conditions* (H.i)–(H.iii) *is* $I(m, n) = O(m^{2/3}n^{2/3} + m + n)$.

REMARKS. (1) Remark (1) after Lemma 2.1 also applies to the bound in Theorem 2.2.

(2) Note that the only way condition (H.ii) is exploited in the above proofs is that more than some constant number of points cannot be incident to more than some constant number of common hyperplanes. Another condition achieving the same goal is

(H.ii′) No four points of P lie in a common 2-flat.

In other words, Theorem 2.2 can also be shown if we replace (H.ii) by (H.ii′).

(3) We state as an open problem to prove any superlinear lower bound for $I(m, n)$.

3. The sphere incidence problem

Using Theorem 2.2 and a fairly standard geometric lifting transform (see, for example, [**8**]), we can derive a good upper bound for an incidence problem involving points in three dimensions. To simplify the notation we use the word *sphere* to either mean a sphere in the common Euclidean sense, or a plane. We define a (*generalized*) *ball* as a closed ball, a closed half-space, or the complement of an open ball in three dimensions. Thus, each sphere bounds two balls and for each ball there is a sphere that bounds it. This slightly nonstandard use of terms will be restricted to the scope of this section.

PROBLEM SPECIFICATION. Let P be a set of m points and S a set of n spheres in three-dimensional Euclidean space satisfying these conditions.

(S.i) No three spheres in S intersect in a common circle or line.
(S.ii) Each sphere $s \in S$ bounds a ball s^+ so that $P \subseteq s^+$.

What is the maximum number of incidences between the points and spheres, where the maximum is taken over all sets P of size m and S of size n satisfying (S.i) and (S.ii)?

We derive an upper bound on the number of incidences by mapping P to a set of m points and S to a set of n hyperplanes in four dimensions so that conditions (H.i)–(H.iii) are satisfied. More specifically, a point $p = (\pi_1, \pi_2, \pi_3)$ is mapped to the point $\bar{p} = (\pi_1, \pi_2, \pi_3, \pi_1^2+\pi_2^2+\pi_3^2)$. Note that $\bar{p}$ is the vertical projection of p, a point in the $x_1x_2x_3$-space, onto the paraboloid of revolution specified by the equation $x_4 = x_1^2 + x_3^2 + x_3^2$. Similarly, a sphere s in the $x_1x_2x_3$-space is mapped to a four-dimensional hyperplane $\bar{s}$ by vertically projecting each of its points onto the same paraboloid and defining $\bar{s}$ as the (unique) hyperplane that contains all these points. The crucial property of this transform is that a point p lies outside, on, or inside a sphere s iff $\bar{p}$ lies vertically above, on, or below $\bar{s}$. (If s is a plane then $\bar{s}$ is a vertical hyperplane (parallel to the x_4-axis) and sidedness is maintained as in the general case.) It thus follows that the sets $\overline{P} = \{\bar{p} | p \in P\}$ and $\overline{S} = \{\bar{s} | s \in S\}$ satisfy condition (H.iii). Condition (H.i) holds because no three points of the paraboloid are collinear, and (H.ii) follows from (S.i). What we said above includes as a special case that $p \in s$ iff $\bar{p} \in \bar{s}$, which implies the following upper bound for the number of point/sphere incidences.

THEOREM 3.1. *The number of incidences between m points and n spheres in three dimensions satisfying conditions* (S.i) *and* (S.ii) *is* $O(m^{2/3}n^{2/3} + m + n)$.

REMARKS. (1) In agreement with remark (1) after Lemma 2.1, condition (S.i) can be relaxed to allow up to some constant number of spheres intersecting a common circle or line.

(2) As noted in remark (2) after Theorem 2.2, it is possible to replace condition (H.ii) by (H.ii′) without sacrificing the $O(m^{2/3}n^{2/3} + m + n)$ upper bound. By the same reason it is possible to replace (S.i) by

(S.i′) No four points of P lie on a common circle or line

without sacrificing the upper bound given in Theorem 3.1.

(3) No superlinear lower bound for the sphere incidence problem of this section is currently known to the authors; see also remark (3) after Theorem 2.2.

4. Combinatorial distance problems

In this section we apply Theorem 3.1 to get upper bounds on problems about repeated distances between points in three-dimensional Euclidean space. Many such problems were originally posed by Paul Erdös and were considered before in the literature. We refer to the problem collection of Moser and Pach [14] as a general source of relevant information.

4.1. Bichromatic minimum distance. Given a set P of m blue points and a set Q of n red points in three-dimensional Euclidean space, what is the maximum number of pairs $(p, q) \in P \times Q$ that realize the minimum distance between points of different color? The best previous bound, derived in [4], is $O(m^{3/4}n^{3/4}\beta(m, n) + m + n)$, with $\beta(m, n) = 2^{\Theta(\alpha(m^3/n)^2)}$ and α the inverse of Ackermann's function. The reduction to Theorem 3.1 should be obvious: around each blue point draw a sphere with radius equal to the minimum bichromatic distance. Condition (S.i) is satisfied because all spheres are equally large, and condition (S.ii) holds because all red points lie on or outside all spheres.

THEOREM 4.1. *The number of bichromatic minimum distance pairs in a set of m blue and n red points in three-dimensional Euclidean space is* $O(m^{2/3}n^{2/3} + m + n)$.

REMARK. In the case of a monochromatic set of m points in three dimensions the maximum number of minimum distance pairs is $\Theta(m)$. This is because if 13 or more points have the same distance to a point p then at least two of them are closer to each other than to p. It follows that each point p belongs to at most 12 minimum distance pairs. This packing argument fails in the bichromatic case because points of the same color can be arbitrarily close to each other.

4.2. Bichromatic nearest neighbors. Given sets P and Q as before, call $(p, q) \in P \times Q$ a *(blue/red) nearest neighbor pair* and q a *nearest neighbor* of p if $q \in Q$ minimizes the Euclidean distance from p to Q. Let $N(p)$ be the number of nearest neighbors of p and consider $\sum_{p \in P} N(p)$. We will assume that either no three points of P are collinear or that no four points of Q are cocircular. The best previous upper bound for this sum is $O(m^{3/4}n^{3/4}\beta(m, n) + m + n)$ (see [4]). By drawing a sphere around each blue point p, with radius equal to the distance between p and its nearest neighbors, we again reduce the distance counting problem to the problem of §3. If no three points of P are collinear we get condition (S.i) and if no four points of Q are cocircular we get condition (S.i′) (see remark (2) after Theorem 3.1). In both cases we obtain the following upper bound.

THEOREM 4.2. *The number of blue/red nearest neighbor pairs in a set of m blue and n red points in three-dimensional Euclidean space is* $O(m^{2/3}n^{2/3} + m + n)$ *if no three blue points are collinear or no four red points are cocircular.*

REMARKS. (1) As in the case of repeated minimum distance pairs, the maximum number of nearest neighbor pairs is $\Theta(m)$ for a monochromatic set of m points in three dimensions. Again, the packing argument used to prove the upper bound fails in the bichromatic case.

(2) The case where all points of Q lie on a circle in three-dimensional space and all points of P lie on the axial line of this circle shows that the bound of Theorem 4.2 does not hold without restrictions on the locations of the points.

4.3. Furthest neighbors. Let P be a (monochromatic) set of m points in three dimensions, and call (p, q) a *furthest neighbor pair* and q a *furthest neighbor* of p if $q \in P$ maximizes the Euclidean distance from p. For each $p \in P$ let $F(p)$ be the number of furthest neighbors and consider $\sum_{p \in P} F(p)$. Assuming no three points are collinear, the best previous bound on this sum can be found in **[4]** and is $O(m^{3/2}\beta(m))$, with $\beta(m) = 2^{\Theta(\alpha(m)^2)}$, improving earlier bounds given in **[6, 9]**. If we draw around each point p the sphere with radius equal to the distance between p and its furthest neighbors we get an instance of the problem in §3. Indeed, condition (S.ii) is satisfied because no point lies outside any of these spheres, and we get condition (S.i) if no three points are collinear and (S.i$'$) if no four points are cocircular. This implies the following result.

THEOREM 4.3. *The number of furthest neighbor pairs in a set of* m *points in three-dimensional Euclidean space is* $O(m^{4/3})$ *if no three points are collinear or no four points are cocircular.*

REMARKS. (1) An example similar to the one in remark (2) after Theorem 4.2 shows that the maximum number of furthest neighbor pairs is quadratic in m if no condition on the location of the points is imposed.

(2) It is worthwhile to note that the maximum number of maximum distance pairs is $\Theta(m)$, namely $2m - 2$, as shown independently in **[12, 13, 16]**.

4.4. Delaunay triangulations. There is a relation between three-dimensional Delaunay triangulations and the incidence problems considered in §§2 and 3. Let P be a set of m points in three dimensions. We call $P' \subseteq P$ a *proper Delaunay subset* if there is a unique sphere so that the points of P' lie on the sphere and all other points of P lie outside the sphere, and we call the sphere a *Delaunay sphere.* The *Delaunay triangulation* of P, denoted by $\mathscr{D}(P)$, is the cell complex whose bounded cells are the convex hulls of the proper Delaunay subsets of P and whose unbounded cell is the complement of the convex hull of P (see **[7]** or **[8]**). We note that $\mathscr{D}(P)$ is not necessarily a triangulation because there may be bounded cells that are not tetrahedra. In fact, the problem we study below is interesting only if many of the cells are not tetrahedral.

Let the *degree* of a cell of $\mathscr{D}(P)$ be its number of vertices; the number of edges of the cell is at most three times the degree minus 6 and the number of 2-faces is at most twice the degree minus 4. We consider the problem of bounding the sum of degrees of subsets of the collection of Delaunay cells.

Let S be the set of all Delaunay spheres. By definition, all points of P lie on or outside any Delaunay sphere. In other words, P and S satisfy condition (S.ii) of §3. We now argue that S also satisfies (S.i), that is, no three Delaunay spheres intersect in a common circle. Suppose to the contrary that there are three Delaunay spheres, s_1, s_2, and s_3, that meet in a common circle. It follows that the centers of the three spheres lie on a common line. Assume that the center of s_2 lies between the other two centers. But then all points of s_2 that do not belong to the common circle lie either inside s_1 or inside s_3. Because no points of P can lie inside s_1 or s_3 it follows that all points of $P \cap s_2$ lie on a circle which contradicts the assumption that s_2 is a Delaunay sphere. Using Theorem 3.1 we thus get the following bound on the sum of degrees. Note that since the Delaunay triangulation of P has at most $O(m^2)$ cells, the term n in the bound of Theorem 3.1 is always subsumed by the leading term.

THEOREM 4.4. *The sum of degrees of n cells in the Delaunay triangulation of any m points in three-dimensional Euclidean space is* $O(m^{2/3}n^{2/3} + m)$.

REMARK. Because of the dual correspondence between Delaunay triangulations and Voronoi diagrams (see [**8**]), Theorem 4.4 implies that the total number of edges incident to n vertices of the Voronoi diagram of any m points in three dimensions is $O(m^{2/3}n^{2/3} + m)$.

5. Conclusions

This paper studies a number of combinatorial distance problems in three-dimensional Euclidean space and derives improved upper bounds for all problems considered. The bounds are direct applications of an $O(m^{2/3}n^{2/3} + m + n)$ bound on the number of incidences between m points and n hyperplanes in four dimensions which holds if no three points are collinear, no three hyperplanes intersect in a common 2-flat, and each hyperplane bounds a closed half-space that contains all points. It is possible to extend the techniques and results of this paper to $d \geq 5$ dimensions. However, because of the many different general position assumptions possible we face a multitude of different problems with different answers. It seems worthwhile to come up with a classification of the different cases to guide future efforts in this direction.

Coincidentally, the above upper bound is the same as for the number of incidences between m points and n lines in the plane, without restriction on the points and lines. However, unlike in the planar problem where the upper bound is known to be tight, there is no superlinear lower bound known for the hyperplane incidence problem. To close the gap between the current upper and lower bounds is the most important open problem suggested by the results of this paper.

We would like to point out that although the results of this paper are mainly combinatorial, the techniques also have algorithmic applications. For example, [**1**] gives a randomized algorithm inspired by our constructive proof that finds a bichromatic minimum distance pair for a set of m blue and red points in three dimensions in expected time $O(m^{4/3}\log^{4/3} m)$. This algorithm is used to construct a minimum spanning tree of m points in three dimensions in the same amount of time.

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