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# SELECTING HEAVILY COVERED POINTS*

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**Abstract.** A collection of geometric *selection lemmas* is proved, such as the following: For any set $P$ of $n$ points in three-dimensional space and any set $\mathcal{S}$ of $m$ spheres, where each sphere passes through a distinct point pair in $P$, there exists a point $x$, not necessarily in $P$, that is enclosed by $\Omega(m^2/(n^2 \log^6 \frac{n^2}{m}))$ of the spheres in $\mathcal{S}$. Similar results apply in arbitrary fixed dimensions, and for geometric bodies other than spheres. The results have applications in reducing the size of geometric structures, such as three-dimensional Delaunay triangulations and Gabriel graphs, by adding extra points to their defining sets.

**Key words.** discrete geometry, computational geometry, selecting points, covering, intervals, boxes, spheres, Delaunay triangulations, finite-element meshes, Gabriel graphs

**AMS subject classifications.** 05B99, 51M99, 52A99, 68Q20, 68R05

**1. Introduction.** The research that led to the results reported in this paper was originally focused on a problem about Delaunay triangulations for finite point sets in three-dimensional space. For such a set $P = \{p_1, p_2, \ldots, p_n\}$, the *Delaunay triangulation*, $\mathcal{D}(P)$, consists of all tetrahedra whose circumscribed spheres enclose no points of $P$ [7], [10], [17]. Depending on how the points are distributed, the number of edges can vary between linear and quadratic in $n$. Euler's relation for three-dimensional cell complexes implies that the number of triangles and tetrahedra, and therefore the total combinatorial size of $\mathcal{D}(P)$, is proportional to the number of edges. We considered the question whether for every set of $n$ points $P$ there exists a point set $Q$ so that $\mathcal{D}(P \cup Q)$ is guaranteed to have only a small number of edges. This question is motivated by the use of Delaunay triangulations in the discretization of three-dimensional objects [4], for finite-element analysis and related applications, where the size of the analysis has a strong effect on the efficiency of the analysis [18]. Of course, any set of $n$ points in three dimensions admits a linear-size triangulation [10]; however, the Delaunay triangulation is preferred in these applications, because its tetrahedra are, in a certain sense, the most "round" possible, a property that affects the quality of the finite-element analysis.

A fairly intuitive approach to the problem is to identify a point that lies inside a large number of spheres circumscribing the tetrahedra of the current Delaunay triangulation. Adding this point will remove all corresponding tetrahedra and replace them by at most a linear number of new tetrahedra. Thus, the problem of slimming Delaunay triangulations can be attacked by showing that if there are many circumscribing spheres then there must be a point enclosed by many of them. It turns out that this is indeed true, for certain quantifications of "many," and

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that similar results can be obtained in more general settings, involving various other geometric objects, in two, three, and beyond three dimensions. We now summarize the main results and present the outline of this paper.

TABLE 1
*Summary of combinatorial results on multiply covered points.*

| objects | dimension | bound | § |
|---|---|---|---|
| intervals | 1 | $\Omega(m^2/n^2)$ | 2.1 |
| rectangular boxes | $d$ | $\Omega\left(m^2 \Big/ \left(n^2 \log^{2d-2} \frac{n^2}{m}\right)\right)$ | 2.2 |
| diameter spheres | $d$ | $\Omega\left(m^2 \Big/ \left(n^2 \log^{2d-2} \frac{n^2}{m}\right)\right)$ | 3.1 |
| general spheres | $d$ | $\Omega\left(m^2 \Big/ \left(n^2 \log^{2d} \frac{n^2}{m}\right)\right)$ | 3.2 |

Sections 2 and 3 present the main results of the paper. They are combinatorial in nature and show how to select multiply covered points in collections of rectangular boxes (§2) and spheres or more general convex bodies (§3). Table 1 lists these results. In each case, the problem is defined for a set of $n$ points in $d$ dimensions, and for a subset of $m$ of the $\binom{n}{2}$ point pairs, where each of these pairs defines a geometric object of some kind. The bound given in the third column of the table is $\Omega(f(n,m))$ if there is always a point enclosed by at least that many of the $m$ objects. In all cases, the bounds are nontrivial only if the number of objects is significantly larger than the number of points.

Sections 4 and 5 discuss the problem of reducing the combinatorial size of certain geometric structures by adding new points. The combinatorial result for general spheres is used in §4 to show, using a constructive proof, that for any set $P$ of $n$ points in three dimensions there is a set $Q$ of $O(n^{1/2}\log^3 n)$ points so that the Delaunay triangulation of $P \cup Q$ has at most $O(n^{3/2}\log^3 n)$ edges. Section 5 studies the case of *Gabriel graphs*. The Gabriel graph of a set $P$ of $n$ points in $d \geq 1$ dimensions, denoted by $\mathcal{G}(P)$, has an edge between two points $p$ and $q$ in $P$ if and only if the sphere whose diameter is $pq$ encloses no point of $P$. We show that the size of $\mathcal{G}(P)$ in three dimensions can be $\Omega(n^2)$, and that it can be slimmed down by adding extra points, as in the case of Delaunay triangulations.

The idea of adding points to slim down the size of Delaunay triangulations has already been used in a paper of Chew [6], where he triangulates polygons without small angles, by finding sharp triangles in the constrained Delaunay triangulation of the polygon, and by adding new points at their circumcenters. After the original appearance of this paper [5], an improved and fairly complete solution to the slimming problem has been given by Bern, Eppstein, and Gilbert [3] (see also [2]), who showed that, in any fixed dimension, $O(n)$ points can always be added to any given set of $n$ points, to reduce the size of the Delaunay triangulation of the combined set to linear in $n$. The technique of [3] is not really comparable to the approach taken here, and it does not supercede our main selection lemmas, which, as we believe, provide useful machinery for tackling other, unrelated geometric problems. Indeed, our selection results have been used in a companion paper [1] to derive an improved bound on the number of halving planes of a point set in three dimensions.

**2. Selecting a point within rectangular boxes.** The primary combinatorial tool used to prove the results of this paper is what we call the "selection lemma" (Lemma 2.1). This section formulates and proves this lemma and demonstrates its generalization to rectangular boxes in

$d \geq 2$ dimensions. Although we phrase the results in geometric terms, they are combinatorial in nature.

**2.1. The selection lemma.** To state the selection lemma we make the following definition. For two points $p < q$ on the real line we call $\beta_{pq} = \{x \mid p < x < q\}$ the *interval* of $\{p, q\}$. For any set $V$, we denote by $\binom{V}{2}$ the set of all unordered pairs $\{p, q\}$, for $p \neq q \in V$. The following lemma can also be found in [1], where generalizations different from the ones in this paper are studied.

LEMMA 2.1. *Let $V$ be a set of $n$ points on the real line and let $E \subseteq \binom{V}{2}$ be a set of $m$ edges. For a point $x$ not necessarily in $V$, let $E(x)$ denote any subset of the edges in $E$ whose intervals contain $x$, define $m(E(x)) = |E(x)|$, and let $n(E(x))$ be the number of points incident to (i.e., endpoints of) edges in $E(x)$.*

(i) *There is a point $x$ and a set $E(x)$ with $m(E(x)) \geq m^2/4n^2$.*

(ii) *There is a point $y$ for which there is a set $E(y)$ with*

$$m(E(y))/n(E(y)) \geq m/\left(6n \log \frac{n^2}{m}\right).$$

*Both bounds are tight up to multiplicative constants.*[1]

*Proof.* We assume that $m \geq 2n$; otherwise both assertions hold trivially. In order to show (i) choose $k - 1$ points, none of which are in $V$, cutting the line into $k$ intervals so that each contains no more than $\lceil \frac{n}{k} \rceil < \frac{n}{k} + 1$ points of $V$ ($k$ will be specified later). The number of edges whose intervals contain none of the $k - 1$ points is therefore at most $k\binom{\lceil \frac{n}{k} \rceil}{2} < (n^2 + nk)/2k$. Each of the remaining intervals contains at least one of the $k - 1$ points and there are at least $m - (n^2 + nk)/2k$ such intervals, which is at least $\frac{m}{2}$ if we choose $k = \lceil n^2/(m - n) \rceil$. By the pigeonhole principle one of the chosen points is contained in at least $m/2(k - 1) \geq (m^2 - mn)/2n^2 \geq m^2/4n^2$ intervals (it is only in the last inequality that we needed the assumption $m \geq 2n$).

It is easy to see that this bound is tight, up to the multiplicative constant. For given $m$ and $n$ let $V$ consist of about $n^2/2m$ groups of about $\frac{2m}{n}$ consecutive points each, and let $E$ contain only edges within but not across groups. Any point $x$ can only be covered by the intervals within one group and there are at most about $m^2/n^2$ such intervals covering a common point.

To prove (ii), build an ordered minimum height binary tree whose nodes are the $k - 1$ chosen points (for the same $k$ chosen in (i)), so that the tree inorder gives the points sorted from left to right. The height of the tree is $h = \lfloor \log(k - 1) \rfloor \leq 2 \log n^2/m$, as is easily verified. For a node $y$ define $E(y)$ as the set of edges in $E$ whose intervals contain $y$ but no ancestors of $y$. In this way each edge whose interval contains at least one of the $k - 1$ points is counted exactly once. By what we said above we therefore have $\sum_y m(E(y)) \geq \frac{m}{2}$. Because each point can be incident to edges of at most one node per level we also have $\sum_y n(E(y)) < n(1 + h)$. Now suppose that $m(E(y))/n(E(y)) < m/(2n(1 + h))$ for each node $y$. But then

$$\sum_y m(E(y)) < \frac{m}{2n(1 + h)} \sum_y n(E(y)) < \frac{m}{2},$$

which is a contradiction. This implies that there is a point $y$ with $m(E(y))/n(E(y)) \geq m/(2n(1 + h)) \geq m/\left(6n \log \frac{n^2}{m}\right)$.

The remainder of the proof shows that the lower bound in (ii) is tight, up to the multiplicative constant. The argument consists of two steps. For the first step consider the graph defined by the set of points $W = \{1, 2, \ldots, \ell\}$ and the set of edges $F = \{\{i, j\} \mid j - i \text{ is a power of } 2\}$.

[1] All logarithms in this paper are to the base 2.

Notice that $|F| = \Theta(\ell \log \ell)$. We show that the edges whose intervals contain some arbitrary point $y$ form a forest by arguing that these edges cannot form a cycle. So assume there is a cycle of edges $\{i_0, i_1\}, \{i_1, i_2\}, \ldots, \{i_k, i_0\}$ whose intervals all contain $y$, and let $i_0$ be the point closest to $y$ (we may assume that $y$ is not an integer multiple of $\frac{1}{2}$ so $i_0$ is uniquely defined). By definition we have $|i_j y| < |i_{j+1} y|$ for $j = 0$ and we now argue that this is true in general. Assume it is true up to $j$. Because $|i_j y| < |i_{j+1} y|$ and the lengths of all intervals are powers of 2, $|i_{j+1} i_{j+2}| \geq 2|i_j i_{j+1}|$ unless $i_{j+2} = i_j$, which is impossible because this would mean that an edge is reused. Consequently, the distances of the $i_j$ from $y$ strictly increase with increasing index, which contradicts the assumption of a cycle. Since every subgraph of a forest is again a forest and since every forest has more vertices than edges the above argument proves that the lower bound in (ii) is asymptotically tight for $m = \Theta(n \log n)$. Nothing has to be proved if $m$ is even smaller than that.

The second step covers other ratios of $m$ and $n$ as follows. For each point $i \in W$ let $V$ contain a group, $G_i$, of $\kappa$ consecutive points, for $\kappa$ some fixed positive integer. We also define $E = \{\{p, q\} \mid p \in G_i,\ q \in G_j,\ \{i, j\} \in F\}$. Now, $n = |V| = \kappa\ell$ and $m = |E| = \Theta(\kappa^2 \ell \log \ell)$ and therefore $\frac{m}{n} = \Theta(\kappa \log \ell)$. We show below that $m(E(y))/n(E(y)) \leq \kappa$ for every point $y$ and every subset $E(y)$ of the set of edges in $E$ whose intervals contain $y$. But this is equivalent to showing that (ii) is asymptotically tight because

$$\frac{m}{6n \log \frac{n^2}{m}} = \Theta\left(\frac{\kappa \log \ell}{\log \frac{\ell}{\log \ell}}\right) = \Theta(\kappa).$$

To show $m(E(y))/n(E(y)) \leq \kappa$ let $E(y)$ be a subset of the edges whose intervals contain $y$ and let $n_i$ be the number of points in $G_i$ incident to at least one edge in $E(y)$. Define $F(y)$ as the set of pairs $\{i, j\} \in F$ so that $E(y)$ contains an edge $\{p, q\}$ with $p \in G_i$ and $q \in G_j$. Clearly, $m(E(y)) = |E(y)| \leq \sum_{\{i,j\} \in F(y)} n_i n_j$. By the argument of the previous paragraph, $F(y)$ defines a forest which implies the existence of a leaf $i$ whose contribution to $\sum n_i n_j$ is therefore at most $n_i \kappa$. Since we can reduce a forest to the empty graph by repeatedly removing a leaf with its incident edge, we get $\sum n_i n_j \leq \kappa \sum n_i = \kappa n(E(y))$, thus proving that (ii) is asymptotically tight. □

*Remarks.* (1) Part (ii) of the selection lemma implies an inequality that is only slightly weaker than (i). To see this note that $m(E(y))/n(E(y))^2 \leq 1$, which implies $n(E(y)) \geq m/\left(6n \log \frac{n^2}{m}\right)$ using (ii). Using (ii) again gives $m(E(y)) \geq m^2/\left(36n^2 \log^2 \frac{n^2}{m}\right)$.

(2) The proofs of the lower bounds in the selection lemma are constructive. Assume the graph $(V, E)$ is given with the points sorted from left to right. Point $x$ can be found in time $O(m)$ by a single scan from left to right that keeps track of how many intervals cover the gap between the current two adjacent points. By a slightly more complicated algorithm we can also find a point $y$ satisfying (ii) in time $O(m)$. The idea is to build explicitly the binary tree described in the proof above (see also [9]). We first build the tree in time $O(k)$ and then assign the endpoints of the edges to the gaps between the $k - 1$ points in time $O(m)$ during a left to right scan. From the gaps of its endpoints we get the leftmost and rightmost of the $k-1$ points that lie in the interval of the edge and we get the lowest common ancestor of the corresponding two nodes, all in constant time (see [14]). It now remains to traverse all nodes of the tree and to select the best one. If the points in $V$ are not presorted then points $x$ and $y$ can be computed in time $O(m + n \log n)$.

**2.2. Rectangular boxes.** For two points $p = (\pi_1, \pi_2, \ldots, \pi_d)$ and $q = (\phi_1, \phi_2, \ldots, \phi_d)$ in $d$ dimensions we define

$$\beta_{pq} = \{x = (\xi_1, \xi_2, \ldots, \xi_d) \mid \pi_i < \xi_i < \phi_i \text{ or } \phi_i < \xi_i < \pi_i \text{ for } 1 \leq i \leq d\}$$

and call it the *box* of $\{p, q\}$. We now generalize Lemma 2.1 from intervals to boxes in $d$ dimensions.

THEOREM 2.2. *Let $V$ be a set of $n$ points in $d \geq 1$ dimensions, so that no two coordinates of any two points in $V$ are the same. Let $E \subseteq \binom{V}{2}$ be a set of $m \geq 2n$ edges. For a point $x$ not necessarily in $V$, let $E(x)$ denote any subset of the edges in $E$ whose boxes contain $x$, define $m(E(x)) = |E(x)|$, and let $n(E(x))$ be the number of points incident to edges in $E(x)$. Then there exists a constant $c_d > 0$ depending only on $d$ such that the following holds.*

(i) *There is a point $x$ and a set $E(x)$ with $m(E(x)) \geq m^2/\left(c_d n^2 \log^{2d-2} \frac{n^2}{m}\right)$.*

(ii) *There is a point $y$ for which there is a set $E(y)$ with*

$$m(E(y))/n(E(y)) \geq m/\left(c_d n \log^d \frac{n^2}{m}\right).$$

*Proof.* We prove the theorem for $c_d = 6^{2^d-1}$ using induction over $d$; the base case, $d = 1$, is settled by the selection lemma. We remark that no effort is made to minimize $c_d$.

If $d \geq 2$ then project all points orthogonally onto the $(d-1)$-dimensional hyperplane $x_d = 0$. By the inductive assumption there is a point $y'$ in this hyperplane and a subset $E(y')$ of the edges in $E$ whose $(d-1)$-dimensional boxes (the projections of the boxes $\beta$) contain $y'$ so that

$$\frac{m(E(y'))}{n(E(y'))} \geq \frac{m}{c_{d-1} n \log^{d-1} \frac{n^2}{m}}.$$

The edges whose $(d-1)$-dimensional boxes contain $y'$ are such that their $d$-dimensional boxes intersect the line parallel to the $d$th coordinate axis that goes through $y'$. On this line we have a one-dimensional problem with $m(E(y'))$ intervals defined by $n(E(y'))$ endpoints. The selection lemma thus implies that there are points $x$ and $y$ with

$$m(E(x)) \geq \frac{m(E(y'))^2}{4n(E(y'))^2} \geq \frac{m^2}{c_d n^2 \log^{2d-2} \frac{n^2}{m}}$$

because $4c_{d-1}^2 \leq c_d$, and

$$\frac{m(E(y))}{n(E(y))} \geq \frac{m(E(y'))}{6n(E(y')) \log \frac{n(E(y'))^2}{m(E(y'))}} \geq \frac{m}{c_d n \log^d \frac{n^2}{m}}$$

because $6c_{d-1} \log(n(E(y'))^2/m(E(y'))) \leq 6c_{d-1} \log((n^2/m) \cdot c_{d-1} \log^{d-1}(n^2/m)) \leq c_d \log^d(n^2/m)$ if $d \geq 2$. □

*Remarks.* (1) Here is a purely combinatorial formulation of Theorem 2.2: Take a graph with vertex set $\{1, 2, \ldots, n\}$ and a set of $m$ edges, and consider $d$ permutations of the vertex set. Then it is possible to cut each permutation into a left and a right part so that there are "many" edges $\{i, j\}$ with $i$ and $j$ separated in each permutation. How many such edges there are is quantified as in Theorem 2.2.

(2) A noninductive proof of Theorem 2.2 can be given by choosing some $k$ points in $d$ dimensions and then using the pigeonhole principle directly. If the point set is based on the so-called $d$-fold rectangle or interval tree [9] then the same bounds as above can be derived.

(3) We have seen that the lower bounds of the (one-dimensional) selection lemma are tight up to the multiplicative constants. This is equivalent to saying that Theorem 2.2 is asymptotically tight for $d = 1$. Are the bounds of Theorem 2.2 asymptotically tight also for $d \geq 2$?

(4) Note that (ii) implies (i) up to a polylogarithmic factor. This is because $m(E(y))/(n(E(y))^2) \leq 1$ and therefore $n(E(y)) \geq m/\left(c_d n \log^d \frac{n^2}{m}\right)$ using (ii). Using (ii) again gives $m(E(y)) \geq m^2/\left(c_d^2 n^2 \log^{2d} \frac{n^2}{m}\right)$.

(5) Given a graph $(V, E)$ with the points sorted along each axis, a point $y$ satisfying Theorem 2.2 (ii) can be computed in time $O(m)$. The algorithm that finds $y$ within this time bound iterates the one-dimensional algorithm mentioned in remark (2) after the selection lemma, once for each dimension. A point $x$ satisfying (i) can be constructed in the same amount of time. If no presorting is assumed then the time to find points $x$ and $y$ is $O(m + n \log n)$.

**3. Selecting a point within spheres.** This section extends the selection lemma to circles, spheres, and other geometric objects. In §3.1 we consider spheres defined by antipodal point pairs. In §3.2 we generalize the result to the case where the sphere defined by two points is arbitrary as long as it passes through the two points. We say that a sphere *encloses* a point, or the point lies *inside* the sphere, if the point belongs to the open ball bounded by the sphere. Section 3.3 studies a sufficient but fairly general condition that allows a similar result as for spheres. Finally, §3.4 presents a curious application of our methods to a problem about points and angles.

**3.1. Diameter spheres.** Let $V$ be a set of $n$ points in $d \geq 2$ dimensions. The *diameter sphere* of a point pair $\{p, q\}$, $\delta_{pq}$, for $p, q \in V$, is the smallest $(d-1)$-sphere that passes through both points. Thus, $z = (p+q)/2$, the midpoint between $p$ and $q$, is its center and $\rho = \frac{|pq|}{2}$, half the distance between $p$ and $q$, is its radius. Observe that for all points $x$ in the box $\beta_{pq}$ the distance to $z$ is smaller than $\rho$. In other words, $\beta_{pq}$ is enclosed in $\delta_{pq}$. Moreover, if we rotate the coordinate axes, as necessary, we may assume that no two coordinates of any two distinct points in $V$ are the same. The following result is therefore an immediate corollary of Theorem 2.2.

COROLLARY 3.1. *Let $V$ be a set of $n$ points in $d \geq 2$ dimensions and let $E \subseteq \binom{V}{2}$ denote any set of $m \geq 2n$ edges. For a point $x$ not necessarily in $V$, let $E(x)$ be a subset of the edges whose diameter spheres enclose $x$, let $m(E(x)) = |E(x)|$, and let $n(E(x))$ be the number of points incident to edges in $E(x)$.*

(i) *There is a point $x$ and a set $E(x)$ with $m(E(x)) \geq m^2/\left(c_d n^2 \log^{2d-2} \frac{n^2}{m}\right)$.*

(ii) *There is a point $y$ for which there is a set $E(y)$ with*

$$m(E(y))/n(E(y)) \geq m/\left(c_d n \log^d \frac{n^2}{m}\right).$$

*Remark.* This result can also be interpreted in terms of angles $\angle pxq$, where $p$ and $q$ are points of $V$ and $x$ is an observation point. We consider all pairs $\{p, q\}$ and thus set $m = \binom{n}{2}$. Point $x$ lies inside $\delta_{pq}$ if and only if $\angle pxq > \frac{\pi}{2}$. Thus, Corollary 3.1 implies that it is always possible to find a point $x$ so that $\Omega(n^2)$ point pairs define an obtuse angle at $x$. Section 3.4 will elaborate on this interpretation and show a similar result for angles larger than $\frac{\pi}{2}$.

**3.2. General spheres.** Next we extend the result for diameter spheres to general spheres. For this extension we let $V$ be a set of $n$ points in $d \geq 2$ dimensions and $E$ be a set of undirected edges between the points as usual. For each edge $\{p, q\} \in E$ we let $\sigma_{pq}$ be an arbitrary but fixed $(d-1)$-sphere that passes through $p$ and $q$. Unless $\sigma_{pq} = \delta_{pq}$, $\sigma_{pq}$ intersects $\delta_{pq}$ in a great-$(d-2)$-sphere of $\delta_{pq}$. Therefore, exactly half of $\delta_{pq}$ is enclosed by $\sigma_{pq}$ and at least half of the ball bounded by $\delta_{pq}$ lies inside $\sigma_{pq}$. If we are lucky then point $x$ (or $y$) of Corollary 3.1 lies in the halves enclosed by the spheres $\sigma$ for a constant fraction of the diameter spheres.

In this case, the bounds of Corollary 3.1 are the same, up to a constant multiplicative factor, as for general spheres. Otherwise, almost all spheres do not contain $x$. We call $\sigma_{pq}$ *anchored* if this is the case, that is, $x$ does not lie inside $\sigma_{pq}$ but it lies inside $\delta_{pq}$. All anchored spheres must lie fairly close to $x$ in the sense that the cone with apex $x$ tangent to any such sphere has opening angle at least $\frac{\pi}{2}$. We will show how to select another point that is guaranteed to lie inside many of the anchored spheres. More precisely, we show the following theorem.

THEOREM 3.2. *Let $V$ be a set of $n$ points in $d \geq 2$ dimensions, and let $E \subseteq \binom{V}{2}$ be a set of $m \geq 2n$ edges. For a point $x$ not necessarily in $V$ let $m(E(x))$ be the number of edges whose spheres enclose $x$.*

(i) *There is a point $x$ with $m(E(x)) \geq m^2 / \left(c''_d n^2 \log^{2d} \frac{n^2}{m}\right)$, where $c''_d$ is a positive constant that depends only on $d$.*

(ii) *There is a point $y$ and a subset $E(y)$ of the edges in $E$ whose spheres enclose $y$ so that*

$$\frac{m(E(y))}{n(E(y))} \geq \frac{m}{c'''_d n \log^{d+1} \frac{n^2}{m}},$$

*where $m(E(y))$ and $n(E(y))$ are defined as usual and $c'''_d$ is some positive constant.*

*Proof.* We prove only (i); claim (ii) can be proved in a similar manner, using Lemma 2.1 (ii) instead of (i). Let $y$ be a point that lies inside many diameter spheres of the edges in $E$, where "many" is quantified as in Corollary 3.1 (ii). Thus, there is a subset $E(y)$ of the edges in $E$ whose diameter spheres enclose $y$ so that

$$\text{(1)} \qquad \frac{m(E(y))}{n(E(y))} \geq \frac{m}{c_d n \log^{d} \frac{n^2}{m}},$$

where $m(E(y)) = |E(y)|$ and $n(E(y))$ is the number of points incident to edges in $E(y)$. Let $\mathcal{S}$ be the set of spheres of edges in $E(y)$ that do not enclose $y$; so all spheres in $\mathcal{S}$ are anchored and we can assume that $|\mathcal{S}| \geq \frac{m(E(y))}{2}$.

To argue about $y$'s view of the world we consider a sphere $\sigma_y$ with center $y$ and centrally project all centers of spheres in $\mathcal{S}$ onto $\sigma_y$. We can assume that no two centers project onto the same point on $\sigma_y$. Define a *cap* of $\sigma_y$ as its intersection with a closed cone with apex $y$ whose opening angle is $\frac{\pi}{6}$, that is, the cone consists of all points $p$ so that the angle between the cone's axis and the half-line through $p$ that starts at $y$ is at most $\frac{\pi}{12}$. By a standard compactness argument, $\sigma_y$ can be covered by a finite (i.e., constant) number, $c'_d$, of caps [12]. Therefore, there exists a cap that contains a constant fraction of the projected centers. Let $R$ be the half-line that is the axis of the corresponding cone $C_R$ and let $\mathcal{S}_R$ be the set of spheres in $\mathcal{S}$ whose centers lie in $C_R$ (that is, project to points in the cap). Since the opening angle of the cone with apex $y$ tangent to any sphere $\sigma$ in $\mathcal{S}_R$ is at least $\frac{\pi}{2}$, it easily follows that $R$ intersects $\sigma$ in two points which delimit an interval that is at least as long as the radius of $\sigma$. To see this it suffices to consider the two-dimensional cross section of $\sigma$ with the plane spanned by $R$ and by the center of $\sigma$. In this plane, the angle $\delta$ between $R$ and the tangent from $y$ to $\sigma$ that is nearer to $R$ (see Fig. 1) is at least $\frac{\pi}{4} - \frac{\pi}{12} = \frac{\pi}{6}$. However, $\delta = \frac{1}{2}s - \frac{1}{2}s'$, where $s$ and $s'$ are the two arcs of $\sigma$, measured in radians, delimited between $R$ and the tangent line. In particular, this implies that the smaller arc cut off $\sigma$ by $R$ is $s + s' \geq \frac{\pi}{3}$, from which it follows trivially that $R$ intersects $\sigma$ in a chord whose length is at least the radius of $\sigma$.

At this point we face a one-dimensional problem on $R$. Intersect $R$ with all open balls bounded by spheres in $\mathcal{S}_R$. This gives a set of at least $m(E(y))/(2c'_d)$ intervals, and we want to show, using the selection lemma, that there is a point in many such intervals and therefore inside many spheres. The difficulty we have to cope with is that the intervals can have many

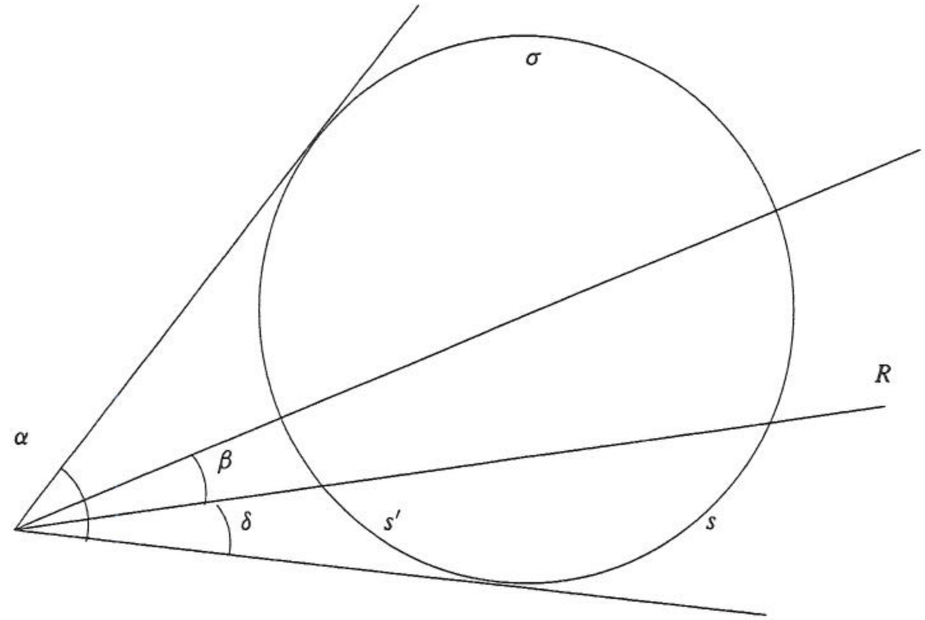


FIG. 1. *R intersects σ in a long chord.*

more than $n(E(y))$ endpoints. In fact, most likely there are twice as many endpoints as there are intervals. We show below that it is possible to replace each interval by an interval contained in it so that the total number of endpoints of the new intervals is at most $6n(E(y))$. Using Lemma 2.1 (i) it follows then that there is a point $x$ contained in

$$m(E(x)) \geq \frac{m(E(y))^2}{4(2c'_d)^2(6n(E(y)))^2}$$

intervals. Together with (1) this implies

$$m(E(x)) \geq \frac{m^2}{c''_d n^2 \log^{2d} \frac{n^2}{m}},$$

where $c''_d = (24c_d c'_d)^2$.

We now show how to reduce the number of endpoints to $6n(E(y))$. Take all spheres in $\mathcal{S}_R$ that go through a common point $p \in V$ and intersect them with the (two-dimensional) plane $h$ that contains $R$ and $p$. Let $\sigma \in \mathcal{S}_R$ go through $p$ and denote by $\bar{\sigma}$ the closed ball bounded by $\sigma$. Clearly, the radius of the circle $h \cap \sigma$ is smaller than or equal to the radius of $\sigma$. Furthermore, the interval $R \cap \bar{\sigma}$ is at least as long as the radius of $\sigma$ because of the way $R$ is chosen. Let $a$ and $b$ be the endpoints of this interval. Then the angle $\angle apb$ is at least $\frac{\pi}{6}$ (see Fig. 2). Hence, 12 half-lines starting at $p$ suffice to stab all these angles, and at most six of them intersect $R$. These at most six half-lines stab all intervals of the form $R \cap \bar{\sigma}_{pq}$ with $\sigma_{pq} \in \mathcal{S}_R$, $p$ fixed, and $q$ arbitrary.

For the final argument we place at most six points for each one of the $n(E(y))$ points incident to edges in $E(y)$, which gives at most $6n(E(y))$ points on $R$. The interval $R \cap \bar{\sigma}_{pq}$ is guaranteed to contain at least one of the at most six points generated by $p$ and at least one of the at most six points generated by $q$. We can thus replace $R \cap \bar{\sigma}_{pq}$ by one of the at most

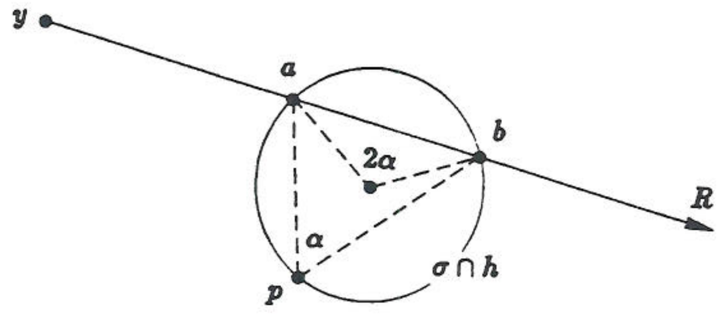


FIG. 2. *The angle formed by a, p, and b is equal to half the angle formed by a, the center of the circle, and b. Since the interval ab is at least as long as the radius of the circle, the latter angle is at least* $\frac{\pi}{3}$.

36 intervals defined by the 12 points generated by $p$ and $q$ and apply the selection lemma as described above. □

*Remark.* The proof of Theorem 3.2 is constructive and leads to an algorithm that computes a point $x$ with the desired properties in time $O(m + n \log n)$. The first step of this algorithm finds a point $y$ within the required number of diameter spheres (see remark (5) after Theorem 2.2). This takes time $O(m + n \log n)$. Second, a ray $R$ that intersects many anchored spheres sufficiently close to their centers is determined by projecting centers of spheres onto the sphere $\sigma_y$ around $y$, covering $\sigma_y$ with a constant number of caps, and choosing the cap that contains the largest number of projected points. This takes time proportional to the number of projected centers, which is $O(m)$. Finally, the spheres whose centers project onto the chosen cap are intersected with $R$, thus the defined intervals are replaced by smaller intervals as described, and point $x$ is selected in time $O(m + n \log n)$ in a single scan along $R$.

**3.3. Round objects.** A result similar to Theorem 3.2 can be established for a more general class of objects than just spheres. Let $p$ and $q$ be two points in $d \geq 2$ dimensions, let $|pq|$ denote their euclidean distance, and let $c_0$ and $C_0$ be two positive constants. A convex set $\tau_{pq}$ is said to be $(c_0, C_0)$-*round* (or simply *round*) for $\{p, q\}$ if

(i) $p$ and $q$ lie on the boundary of $\tau_{pq}$, and
(ii) $\tau_{pq}$ contains a $d$-dimensional ball $\beta_{pq}$ whose radius is at least $c_0|pq|$ and whose center is at a distance at most $C_0|pq|$ from $p$ and from $q$.

For example, the ball bounded by the diameter sphere $\delta_{pq}$ of $p$ and $q$ is $(\frac{1}{2}, \frac{1}{2})$-round, and it is fairly easy to see that any ball with $p$ and $q$ on its boundary is $(\frac{1}{2}, \frac{\sqrt{2}}{2})$-round. With this definition we can show the following generalization of Theorem 3.2.

THEOREM 3.3. *Let $V$ be a set of $n$ points in $d \geq 2$ dimensions and let $E \subseteq \binom{V}{2}$ be a set of $m \geq 2n$ edges $\{p, q\}$, each associated with an round object $\tau_{pq}$. For a point $x$ not necessarily in $V$ let $m(E(x))$ be the number of edges $\{p, q\}$ with $x \in \tau_{pq}$. Then there is a point $x$ with $m(E(x)) \geq m^2/\left(cn^2 \log^{2d} \frac{n^2}{m}\right)$, where $c$ is a positive constant that depends on $d$, $c_0$, and $C_0$.*

*Proof.* To describe where this proof differs from the one of Theorem 3.2 we introduce two auxiliary objects: the ball $\beta'_{pq}$ and the cone $\gamma_{pq}$. The ball $\beta'_{pq}$ has the same center as $\beta_{pq}$ and its radius is half of the radius of $\beta_{pq}$; the cone $\gamma_{pq}$ is the convex hull of $\beta_{pq}$ and $p$ (see Fig. 3). Clearly, we have $\tau_{pq} \supseteq \gamma_{pq} \supseteq \beta_{pq} \supseteq \beta'_{pq}$.

When we construct the half-line $R$ out of point $y$ (defined as in the proof of Theorem 3.2), we make sure it intersects many of the balls $\beta'_{pq}$ associated with edges in $E(y)$. Because of condition (ii), $R$ can be found so that it intersects at least a constant fraction of the $\beta'_{pq}$.

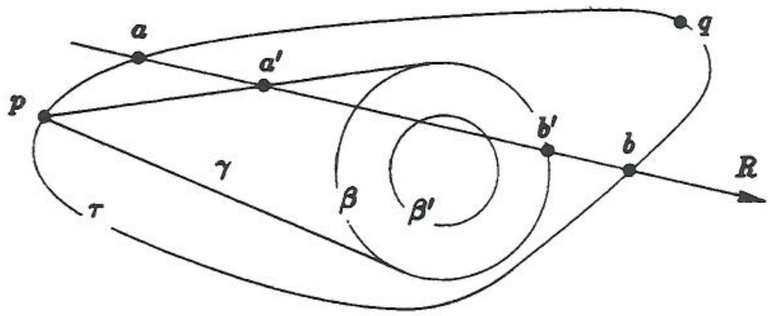


FIG. 3. *The edge* $\{p, q\}$ *defines an round object* $\tau$ *that contains* $\gamma$, $\beta$, *and* $\beta'$. *The half-line* $R$ *intersects* $\beta'$; *its intersection with* $\gamma$ *is* $a'b'$ *and with* $\tau$ *it is* $ab$.

Let us now fix our attention on a particular $\tau = \tau_{pq}$ and let $a$ and $b$ be the endpoints of the interval $R \cap \tau$. In order to complete the proof in the same way as the proof of Theorem 3.2 we need to show that the angle $\angle apb$ (and analogously $\angle aqb$) is at least some constant fraction of $\pi$.

Notice that the boundary of $\gamma = \gamma_{pq}$ consists of a fan of line segments that form the tangents from $p$ to $\beta = \beta_{pq}$, as well as part of the boundary of $\beta$ itself (see Fig. 3). Let $a'$ and $b'$ be the endpoints of $R \cap \gamma$; we will prove the stronger result that the angle $\angle a'pb'$ is at least some fixed fraction of $\pi$. If one of the points $a'$ or $b'$ lies on one of the line segments that form the tangents from $p$ to $\beta$ then the result is immediate: the angle subtended at $p$ goes from the boundary of $\beta$ at least as far as to some point of $\beta'$. By condition (ii) the balls $\beta$ and $\beta'$ look big from $p$, so this angle cannot be too small. On the other hand, if both $a'$ and $b'$ lie on the boundary of $\beta$ then the result follows because $a'b'$ cannot be too short—in particular, it is longer than the radius of $\beta$.

We omit all further details, as they are the same as in the proof of Theorem 3.2. □

*Remarks.* (1) As follows from the above proof, it is not necessary to require that $\tau_{pq}$ be convex and that $p$ and $q$ lie on its boundary. All that is needed is condition (ii) and that $\tau_{pq}$ contains the cones $\gamma_{pq}$ and $\gamma_{qp}$ defined by $\beta_{pq}$ and points $p$ and $q$.

(2) It is also interesting to observe that condition (ii) is not sufficient to prove Theorem 3.3. Indeed a counterexample exists already in one dimension. Let $V = \{p_i = 2^i \mid 1 \le i \le n\}$ be the set of $n$ points and for $i < j$ define $\tau_{ij} = \{x \mid (2p_i + p_j)/3 < x < (p_i + 2p_j)/3\}$. Thus, $\tau_{ij}$ has the same midpoint as the interval $\beta_{ij}$ delimited by $p_i$ and $p_j$ and its length is one third of that of $\beta_{ij}$. However, for any $i < j < k$ we have $\tau_{ij} \cap \tau_{ik} = \emptyset$ because $(2^i + 2^{j+1})/3 < (2^{i+1} + 2^k)/3$. Thus, the set of $\binom{n}{2}$ intervals $\tau_{ij}$ can be partitioned into $n$ subsets so that two intervals are disjoint if they belong to the same subset. It follows that there is no point $x$ contained in more than $n$ intervals $\tau_{ij}$.

**3.4. A problem about points and angles.** For points $p$ and $q$ in $d$-dimensional space and for angle $\alpha$, $\frac{\pi}{2} \le \alpha \le \pi$, define the $\alpha$-*football* of $\{p, q\}$ as the set

$$\phi_{pq}(\alpha) = \{x \mid \angle pxq \ge \alpha\}.$$

For example, $\phi_{pq}(\frac{\pi}{2})$ is the closed ball bounded by the diameter sphere $\delta_{pq}$, and $\phi_{pq}(\pi)$ is the line segment $pq$. For general $\alpha$, $\phi_{pq}(\alpha)$ is the intersection of all closed balls that contain $p$ and $q$ and have a fixed radius depending on $|pq|$ and $\alpha$. If $\alpha < \pi$ is fixed, then $\phi_{pq}(\alpha)$ contains a ball centered at the midpoint between $p$ and $q$ whose radius is some fixed positive fraction of $|pq|$. Hence, $\phi_{pq}(\alpha)$ is round for $C_0 = \frac{1}{2}$ and $c_0 > 0$ ($c_0$ goes to zero if $\alpha$ approaches $\pi$) and Theorem 3.3 applies. We reformulate this result for the case where *every* pair of points defines an $\alpha$-football and phrases it in terms of angles.

COROLLARY 3.4. *Let $P$ be a set of $n$ points in $d \geq 2$ dimensions and let $\alpha \geq \frac{\pi}{2}$ be a fixed angle strictly smaller than $\pi$. Then there exists a constant $c$ depending on $d$ and $\alpha$ and a point $x$ so that $\angle pxq \geq \alpha$ for at least $cn^2$ pairs $\{p, q\} \in \binom{P}{2}$.*

Loosely speaking, point $x$ is almost collinear with a constant fraction of the point pairs if $\alpha$ is insignificantly smaller than $\pi$, for example $\alpha = 179°$. In other words, $x$ almost lies on each one of a constant fraction of the lines defined by the points.

**4. Slimming down spatial Delaunay triangulations.** This section deals with Delaunay triangulations for point sets in (three-dimensional) space. Let $P$ be a set of $n$ points in space and let $\mathcal{D}(P)$ be its Delaunay triangulation. For simplicity we assume that no five points are cospherical so that $\mathcal{D}(P)$ is uniquely defined. If this is not the case then it is always possible to enforce it by simulating an arbitrarily small perturbation of the points; see [11]. As mentioned in the introduction, $abcd$ is a tetrahedron of $\mathcal{D}(P)$ if and only if the sphere through points $a$, $b$, $c$, and $d$ does not enclose any points of $P$.

For $0 \leq i \leq 3$, let $f_i$ be the number of $i$-dimensional faces of $\mathcal{D}(P)$, that is, $f_0 = n$ is the number of vertices, $f_1$ is the number of edges, $f_2$ is the number of triangles, and $f_3$ is the number of tetrahedra of $\mathcal{D}(P)$. By Euler's relation we have $f_0 - f_1 + f_2 - f_3 = 1$ (see Hopf [15] for an elementary proof of this relation). Because every tetrahedron is bounded by four triangles and every triangle bounds at most two tetrahedra we also have $2f_3 \leq f_2$. This implies

$$f_3 \leq f_1 - n + 1 \text{ and } f_2 \leq 2f_1 - 2n + 2. \tag{2}$$

We thus see that $f_1$, the number of edges of $\mathcal{D}(P)$, is a good measure of the combinatorial complexity of $\mathcal{D}(P)$. We call $f_1$ the *size* of $\mathcal{D}(P)$.

Depending on how the points are distributed, the size of $\mathcal{D}(P)$ can vary between linear in $n$ and quadratic in $n$. An extreme example is when the points of $P$ lie on the positive branch of the moment curve, $\mathcal{M} = \{(x, x^2, x^3) \mid x > 0\}$. Because a sphere intersects $\mathcal{M}$ in at most four points, which can be shown using Descartes' sign rule for the polynomial that arises, every point pair defines an edge of $\mathcal{D}(P)$. It follows that the size of $\mathcal{D}(P)$ is $\binom{n}{2}$ (see also [8]). The goal of this section is to show that no matter how badly $P$ is distributed, there is always a small set $Q$ of points in space so that $\mathcal{D}(P \cup Q)$ has size at most $O(n^{3/2} \log^3 n)$.

A sphere is called a *Delaunay sphere* of $P$ if it is the circumscribed sphere of a tetrahedron $abcd$ of $\mathcal{D}(P)$. Using Theorem 3.2 we can show that if there are many Delaunay spheres, then there are many that enclose a common point.

LEMMA 4.1. *Let $P$ be a set of $n$ points in space defining $t$ Delaunay spheres. There is a point $x$ enclosed by $m(E(x)) \geq t^2 / \left(cn^2 \log^6 \frac{n^2}{t}\right)$ Delaunay spheres, for some positive constant $c$.*

*Proof.* Note that an edge $ab$ is incident to as many Delaunay spheres as there are tetrahedra in $\mathcal{D}(P)$ that share $ab$; this number can be as large as $n - 2$. In order to apply Theorem 3.2 we match the edges of $\mathcal{D}(P)$ with the Delaunay spheres so that the Delaunay sphere matched with an edge passes through its endpoints and at least $\frac{t}{6}$ Delaunay spheres have a matching edge. We do this as follows. By definition, each Delaunay sphere is incident to six edges, and, by (2), there are at least $t$ edges. Match an edge with an incident sphere arbitrarily and remove both from further consideration. Thus, there are at most five more edges that can no longer find a matching sphere. If we iterate this process we get at least $\frac{t}{6}$ matched pairs as required.

We thus arrive at a situation where we have $n$ points and $m \geq \frac{t}{6}$ edges with a unique corresponding sphere each. Theorem 3.2 implies that there is a point $x$ enclosed by at least

$$\frac{m^2}{c_3'' n^2 \log^6 \frac{n^2}{m}} \geq \frac{t^2}{6^2 c_3'' n^2 \log^6 \frac{6n^2}{t}} \geq \frac{t^2}{cn^2 \log^6 \frac{n^2}{t}}$$

spheres, e.g., for $c = 6^8 c_3''$. □

If we add $x$ to $P$ then all tetrahedra whose circumscribed spheres enclose $x$ disappear by definition. Lemma 4.1 thus implies that it is possible to destroy $\Omega\left(t^2/\left(n^2 \log \frac{n^2}{t}\right)\right)$ tetrahedra at once. However, $x$ also gives rise to new tetrahedra. Because all new tetrahedra are incident to $x$ we can bound their number from above as follows.

LEMMA 4.2. *Let $P$ be a set of $n$ points in space and $x$ a point not in $P$. Then $x$ is incident to at most $2n - 4$ tetrahedra in $\mathcal{D}(P \cup \{x\})$.*

*Proof.* Let $\sigma$ be a sufficiently small sphere with center at $x$. If we intersect $\sigma$ with the edges, triangles, and tetrahedra of $\mathcal{D}(P)$ we get a planar graph. Each vertex of this graph corresponds to an edge of $\mathcal{D}(P)$, and if $\sigma$ is sufficiently small all such edges are incident to $x$. Because $x$ is incident to at most $n$ edges (at most one per point in $P$), the planar graph has at most $n$ vertices and, by Euler's relation, at most $2n - 4$ regions. These regions correspond to the tetrahedra incident to $x$. □

What we said about Delaunay triangulations in space suggests the following algorithm for reducing the size of a Delaunay triangulation by adding points at well-chosen locations. Recall that $m(E(x))$ is the number of Delaunay spheres destroyed by adding point $x$.

**Input.** A set $P$ of $n$ points in space.
**Output.** A set $Q$ of points in space so that $\mathcal{D}(P \cup Q)$ has at most $O(n^{3/2} \log^3 n)$ edges.
**Algorithm.**

Construct $\mathcal{D}(P)$ and set $Q := \emptyset$;
**loop** find a point $x$ that maximizes $m(E(x))$ in $\mathcal{D}(P \cup Q)$;
  **if** $m(E(x)) \geq 4n$ **then** $Q := Q \cup \{x\}$ and update $\mathcal{D}(P \cup Q)$ accordingly
    **else exit**
  **endif**
**forever.**

Using Lemmas 4.1 and 4.2, one can establish the following result.

THEOREM 4.3. *For any set of $n$ points $P$ in three-dimensional space there is a set $Q$ of at most $O(n^{1/2} \log^3 n)$ points so that the Delaunay triangulation of $P \cup Q$ has at most $O(n^{3/2} \log^3 n)$ edges. Such a set $Q$ can be computed in time $O(n^2 \log^7 n)$.*

We omit here details of the analysis, because this result is less significant now, in view of the recent results of Bern et al. [3]. Interested readers are referred to an earlier and fuller version of this paper [5].

**5. The size of Gabriel graphs.** The Gabriel graph of a finite point set is a subgraph of the Delaunay triangulation that has applications in zoology and geography [13], [16]. Let $P$ be a set of $n$ points in $d \geq 1$ dimensions. The Gabriel graph of $P$, denoted by $\mathcal{G}(P)$, has an edge between two points $p$ and $q$ in $P$ if and only if their diameter sphere, $\delta_{pq}$, encloses no point of $P$. The definition implies that the edges of $\mathcal{G}(P)$ are a subset of the edges of the Delaunay triangulation. Thus, $\mathcal{G}(P)$ has only $O(n)$ edges when $d \leq 2$, and trivially at most $O(n^2)$ edges, otherwise. The bound is tight for $d \leq 2$, since each point is incident to at least one edge. The following lemma shows that the bound is also tight for $d > 2$.

LEMMA 5.1. *The maximum number of edges of the Gabriel graph of n points in $d \geq 3$ dimensions is $\Omega(n^2)$.*

*Proof.* We exhibit a set $P$ of $2n$ points in three dimensions such that $\mathcal{G}(P)$ has at least $n^2$ edges. Embedding this example in higher-dimensional space proves the lemma for $d > 3$.

We place the points in two groups $\{a_i\}$ and $\{b_j\}$ on interlocking, orthogonal circles. Each circle passes through the center of the other, and the points on each circle are located near the center of the other circle. Each circle has radius 2. The points $a_i$ lie near $(0, 1, 0)$ on a circle in the $xy$-plane centered on $(0, -1, 0)$. The $b_j$ lie near $(0, -1, 0)$ on a circle in the $yz$-plane centered on $(0, 1, 0)$. To quantify "nearness" we use a small parameter $\epsilon$:

$$a_i = (f(i), 1 - i\epsilon, 0) \quad \text{and} \quad b_j = (0, -1 + j\epsilon, f(j)),$$

where $1 \leq i, j \leq n$ and $f(k) = \sqrt{4k\epsilon - k^2\epsilon^2} < \sqrt{4k\epsilon}$. We show that for $\epsilon > 0$ sufficiently small, the diameter sphere determined by a pair $\{a_i, b_j\}$ contains no other points of $P$. The center of the sphere is

$$c_{ij} = \frac{a_i + b_j}{2} = \frac{1}{2}(f(i), \epsilon(j - i), f(j)).$$

We prove that the distance from $c_{ij}$ to a point $a_k$ (or $b_k$) is minimized when $k = i$ ($k = j$). The square of the distance is

$$\begin{aligned}(a_k - c_{ij})^2 &= \frac{1}{4}((2f(k) - f(i))^2 + (2 - 2k\epsilon - j\epsilon + i\epsilon)^2 + f(j)^2)\\ &= \frac{1}{4}(16k\epsilon - 4f(k)f(i) + 4i\epsilon + 4 - 8k\epsilon - 4j\epsilon + 4i\epsilon + 4j\epsilon) + O(\epsilon^2)\\ &= 1 + 2\epsilon(k + i) - f(k)f(i) + O(\epsilon^2).\end{aligned}$$

Because $f(k)f(i) = 4\epsilon\sqrt{ki} + O(\epsilon^2)$, we have

$$(a_k - c_{ij})^2 = 1 + 2\epsilon(k - 2\sqrt{ki} + i) + O(\epsilon^2) = 1 + 2\epsilon(\sqrt{k} - \sqrt{i})^2 + O(\epsilon^2).$$

For $\epsilon$ small enough, this quantity is minimized only when $k = i$. □

We can use Corollary 3.1 to reduce the size of Gabriel graphs. In three dimensions this gives a better bound than the one for Delaunay triangulations, which is based on Theorem 3.2.

THEOREM 5.2. *For any set of n points P in $d \geq 3$ dimensions there is a set Q of $O(n^{1/2} \log^{d-1} n)$ points so that the Gabriel graph of $P \cup Q$ has at most $O(n^{3/2} \log^{d-1} n)$ edges.*

*Proof.* Here is a sketch of the proof. By Corollary 3.1, if $m \geq 2n$, $m$ the number of edges of $\mathcal{G}(P \cup Q)$, then there is a point $x$ whose addition to $Q$ deletes $m(E(x))$ edges from $\mathcal{G}(P \cup Q)$, where

$$m(E(x)) \geq \frac{m^2}{c_d n^2 \log^{2d-2} \frac{n^2}{m}}.$$

Adding a point to $Q$ adds at most $|P \cup Q|$ edges to $\mathcal{G}(P \cup Q)$. Using an argument similar to that of §4, one can show that the number of edges of $\mathcal{G}(P \cup Q)$ can be reduced to $O(n^{3/2} \log^{d-1} n)$ by adding points to $Q$. By reasoning similar to that used in the proof of Theorem 4.3, one can show that the algorithm of §4, modified for Gabriel graphs, produces a set $Q$ of size $O(n^{1/2} \log^{d-1} n)$. □

## REFERENCES


[1] B. Aronov, B. Chazelle, H. Edelsbrunner, L. J. Guibas, M. Sharir and R. Wenger, *Points and triangles in the plane and halving planes in space*, Discrete Comput. Geom., 6 (1991), pp. 435–442.

[2] M. Bern and D. Eppstein, *Mesh generation and optimal triangulation*, in Computing in Euclidean Geometry, D.-Z. Du and F. K. Hwang, eds., World Scientific, Singapore, 1992, pp. 23–90.

[3] M. Bern, D. Eppstein, and J. Gilbert, *Provably good mesh generation*, in Proc. 31st IEEE Symp. on Foundations of Computer Science, 1990, pp. 231–241; J. Comp. Systems Science, to appear.

[4] J. C. Cavendish, D. A. Field, and W. H. Frey, *An approach to automatic three-dimensional finite element mesh generation*, Internat. J. Numer. Methods Engrg., 21 (1985), pp. 329–347.

[5] B. Chazelle, H. Edelsbrunner, L. J. Guibas, J. Hershberger, R. Seidel and M. Sharir, *Slimming down by adding; selecting heavily covered points*, Tech Report UIUCDCS-R-90-1574, Department of Computer Science, University of Illinois at Urbana-Champaign, Illinois, March 1990.

[6] L. P. Chew, *Guaranteed-quality triangular meshes*, Tech. Report TR-89-983, Department of Computer Science, Cornell University, Ithaca, NY, 1989.

[7] B. Delaunay, *Sur la sphère vide*, Izvestia Akademii Nauk SSSR, VII Seria, Otdelenie Matematicheskii i Estestvennyka Nauk, 7 (1934), pp. 793–800.

[8] A. K. Dewdney and J. K. Vranch, *A convex partition of $R^3$ with applications to Crum's problem and Knuth's post-office problem*, Utilitas Math., 12 (1977), pp. 192–199.

[9] H. Edelsbrunner, *A new approach to rectangle intersections, part* I, Intern. J. Comput. Math., 13 (1983), pp. 209–219.

[10] ———, *Algorithms in Combinatorial Geometry*, Springer-Verlag, Heidelberg, Germany, 1987.

[11] H. Edelsbrunner and E. P. Mücke, *Simulation of Simplicity: a technique to cope with degenerate cases in geometric algorithms*, ACM Trans. Graphics, 9 (1990), pp. 66–104.

[12] L. Fejes Tóth, *Lagerungen in der Ebene, auf der Kugel und im Raum*, second ed., Springer-Verlag, Berlin, 1972.

[13] K. R. Gabriel and R. R. Sokal, *A new statistical approach to geographic variation analysis*, Systematic Zoology, 18 (1969), pp. 259–278.

[14] D. Harel and R. E. Tarjan, *Fast algorithms for finding nearest common ancestors*, SIAM J. Comput., 13 (1984), pp. 338–355.

[15] H. Hopf, *Über Zusammenhänge zwischen Topologie und Metrik im Rahmen der elementaren Geometrie*, Mathematisch-Physikalische Semester Berichte, 3 (1953), pp. 16–29.

[16] D. W. Matula and R. R. Sokal, *Properties of Gabriel graphs relevant to geographic variation research and clustering of points in the plane*, Geographical Analysis, 12 (1980), pp. 205–222.

[17] F. P. Preparata and M. I. Shamos, *Computational Geometry—an Introduction*, Springer-Verlag, New York, 1985.

[18] G. Strang and G. J. Fix, *An Analysis of the Finite Element Method*, Prentice Hall, Englewood Cliffs, NJ, 1973.