Algebraic Decomposition of Non-convex Polyhedra*

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Abstract

Any arbitrary polyhedron $P \subseteq \mathbf{R}^d$ can be written as algebraic sum of simple terms, each an integer multiple of the intersection of d or fewer half-spaces defined by facets of P. P can be non-convex and can have holes of any kind. Among the consequences of this result are a short boolean formula for P, a fast parallel algorithm for point classification, and a new proof of the Gram-Sommerville angle relation.

1 Introduction

Polyhedra model common objects in our everyday life by piecewise linear approximation. It is important not to be restricted to *convex* polyhedra because most useful objects are indeed non-convex. A spoon has a depression, a drawer is hollow if closed, a coffee cup has a handle, and a pair of chopsticks is not even connected. No convex object can share any of these features, and they are all functionally essential.

Representations of polyhedra. Within mathematics, polyhedra are studied in various areas of geometry and topology. Computational aspects of polyhedra are considered in areas of computer science, including computational geometry and solid modeling. Particular attention is paid to the questions of representation and computational complexity. The former question is fundamental since the latter can only be addressed under an assumed data structure. As it turns out, the complexity of a particular task or function can depend heavily on the available representation. Furthermore, no single representation is best suited for all functions of practical relevance, which brings up the issue of conversion between different data structures. The currently possibly most popular data structure is the *boundary representation*, that specifies faces and their adjacencies. Apparently the only contender in popularity is the *CSG* or *constructive solid geometry representation*, that writes a polyhedron as a boolean expression of half-spaces.

Algebraic decomposition. Boolean expressions can be generalized by replacing the boolean with other algebraic structures. This paper considers the ring of integers and expressions of the form

$$\iota(x) \;=\; \sum_i a_i \cdot 1_{Q_i}(x),$$

where the a_i are integers and the Q_i are open polyhedral cones. x is any point in $\mathbf{R}^d$, and $1_{Q_i} : \mathbf{R}^d \to \{0, 1\}$ is the *indicator* or *characteristic function* of $Q_i \subseteq \mathbf{R}^d$, that is,

$$1_{Q_i}(x) \;=\; \begin{cases} 1 & \text{if } x \in Q_i, \text{ and} \\ 0 & \text{if } x \notin Q_i. \end{cases}$$

We call ι an *algebraic decomposition* of a polyhedron $P \subseteq \mathbf{R}^d$ if $\iota(x) = 1_P(x)$ for all $x \in \mathbf{R}^d$.

The main result of this paper is the construction of an algebraic decomposition of P from the boundary representation. In the intuitive and mathematically more appealing first form of the result, each term $a_i \cdot 1_{Q_i}$ corresponds to a face φ_i of P, and Q_i is the open face figure of φ_i reflected though φ_i. In the computationally more convenient second form, each open face figure Q_i

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is further decomposed into

$$1_{Q_i}(x) = \sum_j a_j \cdot 1_{\cap H_j}(x),$$

where each H_j is a collection of at most d open half-spaces. By taking integers modulo 2 we obtain the third form of the result:

$$P = \bigoplus_{\ell} \bigcap H_\ell,$$

where P is written as a chain of symmetric differences of convex polyhedral cones.

Related work. The conversion from the boundary of a polyhedron to a boolean expression of half-spaces has been considered by Peterson [19], see also [11]. The 2-dimensional case has been solved satisfactorily in [5], but many questions about the most important 3-dimensional case remain unanswered. Indicator functions for polygons in the plane and polyhedra in space have been discussed by Franklin [9]. His formulas sum real weights determined by angles at vertices and are thus of different type than the integer formulas studied in this paper. Straightforward algebraic decompositions into cones of the origin over all faces of a polyhedron are commonly used in solid modeling [14]. These require half-spaces not defined by facets of the polyhedron, and robust implementations of algorithms based on such representations are difficult.

In the mathematics literature, work related to this paper is abundant and goes back several decades. Noteworthy is Hadwiger's extensive study of polyhedra documented in many papers. Much of this work is also covered in his text on volume, surface area, and isoperimetries [12] and in the recent book by Schneider on the Brunn-Minkowski theory [24]. Valuations offer a general framework for algebraic decompositions. We refer to McMullen [15] for a recent survey article on this topic. Algebraic decompositions of polyhedra similar to but different from the ones in this paper have been described by Chen [3] in connection to various discrete notions of curvature. Indicator functions for unions of spherical balls have been studied by Naiman and Wynn [18] and by the author [7]. Their formulas are derived from an algebraic decomposition of *convex* polyhedra, see also [4], which forms the starting point for the work reported in this paper.

Outline. Sections 2 and 3 explain and illustrate the main result of this paper for polygons in $\mathbb{R}^2$ and for polyhedra in $\mathbb{R}^3$. Section 4 develops d-dimensional concepts and notation. Section 5 states the first form of our main result. The proof is rather technical and omitted in the conference version of this paper. Section 6 derives the second and third forms of the result. Section 7 discusses applications of this work.

2 Polygons in the Plane

Indicator functions for non-convex polyhedra are non-trivial even in 2 dimensions. This section specializes the main result of this paper to the plane and avoids most of the technical discussions required in higher dimensions.

Polygons and boundary. Define a *convex polygon* in $\mathbb{R}^2$ as the common intersection of finitely many closed half-planes. A *polygon*, P, is the union of finitely many convex polygons, see figure 1. The boundary of P consists of *edges* and *vertices*. We assume that each vertex is endpoint of only 2 edges, or more formally that P is a 2-dimensional manifold with boundary. A vertex is either *convex* (labeled '+') or *concave* (labeled '−' in figure 1). In the former case, it is a vertex of one of the convex polygons. In the latter case, it is at the intersection of edges from at least two different convex polygons.

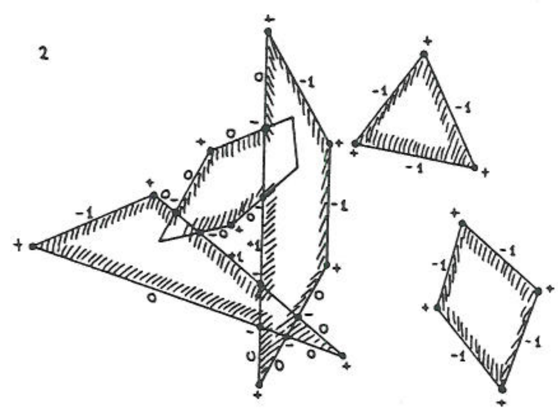

Figure 1: A polygon with three components and one hole. The numbers and signs labeling edges and vertices will be explained at the end of this section.

For an edge, ϵ, let aff ϵ be the line that contains ϵ, and let the *edge figure* be the closed half-plane ff ϵ bounded by aff ϵ and locally on the same side as P. The *vertex figure* of a vertex ν which is endpoint of edges ϵ_1 and ϵ_2 is

$$\text{ff}\,\nu = \begin{cases} \text{ff}\,\epsilon_1 \cap \text{ff}\,\epsilon_2 & \text{if } \nu \text{ is convex, and} \\ \text{ff}\,\epsilon_1 \cup \text{ff}\,\epsilon_2 & \text{if } \nu \text{ is concave.} \end{cases}$$

Negative face figures. We wish to construct an algebraic decomposition of P from local information about edges and vertices. To this end it is more convenient to work with reflections of figures.

The *negative edge figure* of ϵ is $\mathrm{nff}\,\epsilon = 2y - \mathrm{int}\,\mathrm{ff}\,\epsilon$, where y is any point of ϵ. Intuitively, $\mathrm{nff}\,\epsilon$ is the reflection of the open edge figure, $\mathrm{int}\,\mathrm{ff}\,\epsilon$, through ϵ. Similarly, the *negative vertex figure* of ν is $\mathrm{nff}\,\nu = 2\nu - \mathrm{int}\,\mathrm{ff}\,\nu$. Observe that a point x belongs to a negative vertex figure, $\mathrm{nff}\,\nu$, iff the ray from x towards ν enters the interior of P as it passes through ν. A similar and more obvious interpretation holds for negative edge figures. The *indicator function* of a negative edge or vertex figure is denoted by 1_{nff}. With this notation, we specify

$$\Gamma_P(x) \;=\; \chi(P) - \sum_{\epsilon} 1_{\mathrm{nff}\,\epsilon}(x) + \sum_{\nu} 1_{\mathrm{nff}\,\nu}(x), \quad (1)$$

where $\chi(P)$ is the number of components of P minus the number of holes. By the main theorem, $\Gamma_P = 1_P$, and Γ_P is therefore an algebraic decomposition of P.

Half-planes and angles. The formula in (1) can be transformed so each term checks inclusion of x in a single half-plane or the intersection of 2 half-planes. Rewrite the indicator function for every concave vertex ν with edges ϵ_1 and ϵ_2 as

$$1_{\mathrm{nff}\,\nu} \;=\; 1_{\mathrm{nff}\,\epsilon_1} + 1_{\mathrm{nff}\,\epsilon_2} - 1_{\mathrm{nff}\,\epsilon_1} \cdot 1_{\mathrm{nff}\,\epsilon_2}.$$

In effect, this cancels the original contributions of ϵ_1 and ϵ_2 and changes the contribution of ν from a positive union to a negative intersection of 2 half-planes. In the resulting formula the contribution of an edge, ϵ, is -1, 0, or $+1$ times $1_{\mathrm{nff}\,\epsilon}$, depending on whether both endpoints are convex, one is convex and the other is concave, or both are concave. The contribution of a vertex, ν, with edges ϵ_1 and ϵ_2 is $+1$ or -1 times $1_{\mathrm{nff}\,\epsilon_1} \cdot 1_{\mathrm{nff}\,\epsilon_2}$, depending on whether ν is convex or concave.

For an example consider figure 1, where edges and vertices are labeled with their respective coefficients. To improve readability, the vertex labels are abbreviated to $+$ and $-$. The coefficient of P is 2, namely 3 components minus 1 hole.

3 Polyhedra in Space

This section extends the informal discussion of algebraic decompositions from 2 to 3 dimensions. Polyhedra in $\mathbb{R}^3$ are sufficiently more intricate than polygons in $\mathbb{R}^2$, and they provide opportunity to illustrate subtleties complicating the general d-dimensional study.

Polyhedra and boundary. In analogy to the 2-dimensional case, we define a *convex polyhedron* in $\mathbb{R}^3$ as the common intersection of finitely many closed half-spaces, and a *polyhedron*, P, as the union of finitely many convex polyhedra. The boundary of P consists of *vertices*, *edges*, and (2-dimensional) *facets*. We assume P is a 3-dimensional manifold with boundary, and each facet is a 2-dimensional manifold with boundary. Intuitively, this means P is not pinched and neither is any of its facets. The formal definition is given in section 4. Examples of such polyhedra are shown in figures 2 and 3.

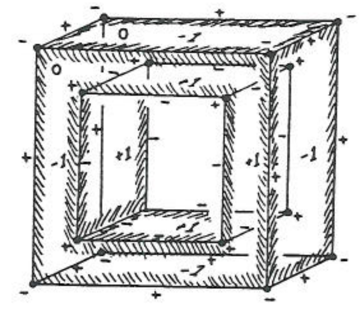

Figure 2: "Rectangular frame" or "box with tunnel". There are 10 facets, 8 rectangular and the front and the back facets with a hole each. The labels indicate the coefficients in the formula with half-space, wedge, and cone terms, as explained at the end of this section. The Euler characteristic of the box with tunnel is 0, namely 1 component minus 1 tunnel plus 0 voids.

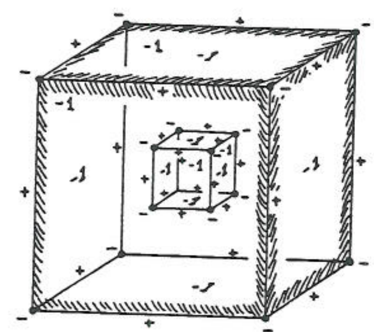

Figure 3: "Cube with cubical void". There are 12 square facets, 6 on the inside and 6 on the outside. The labels carry the same meaning as in figure 2. The Euler characteristic of the cube with void is 2, namely 1 component minus 0 tunnels plus 1 void.

Possible ambiguities in the decomposition of the boundary into faces will be formally resolved in section 4. The notion of face adopted in this paper is sufficiently general to allow facets with holes. Even disconnected facets are admitted, although it is a matter of taste whether two facets in a common plane should be considered different facets or different components of one facet. The same indicator function will work under both interpretations. The most important property to keep in mind in decomposing the boundary is that the collection of vertices, edges, and facets form a complex. Specifically, the intersection of two faces is the union of other faces (possibly the empty union) and the boundary of every face is the union of other faces.

Face figures. *Vertex*, *edge*, and *facet figures* are defined as in the planar case. For a facet ϕ, ff ϕ is always a closed half-space, and for an edge ϵ, ff ϵ is the intersection or union of 2 closed half-spaces. In the former case ϵ is *convex*, and in the latter case it is *concave*. For vertices the situation is more complicated, and simple cases are discussed shortly.

The *negative face figure* of a face φ is $\text{nff}\,\varphi = 2y - \text{int}\,\text{ff}\,\varphi$, where y is any point in φ. This definition applies to vertices, edges, and facets. The *indicator function* of a negative face figure is denoted 1_{nff}. With this notation, we specify

$$\begin{aligned}\Gamma_P(x) &= \chi(P) \\ &- \sum_{\phi} \chi(\phi) \cdot 1_{\text{nff}\,\phi}(x) \\ &+ \sum_{\epsilon} \chi(\epsilon) \cdot 1_{\text{nff}\,\epsilon}(x) \\ &- \sum_{\nu} \chi(\nu) \cdot 1_{\text{nff}\,\nu}(x).\end{aligned}$$

The Euler characteristic of every vertex is $\chi(\nu) = 1$. Similarly, the Euler characteristic of every edge is $\chi(\epsilon) = 1$, unless edges with disconnected components are allowed. In the latter case, each component must belong to the same two facets, see section 4. The Euler characteristic of ϕ is the number of components minus the number of holes. The Euler characteristic of P itself is the number of components minus the number of tunnels (as in figure 2) plus the number of voids (as in figure 3). By the main theorem, $\Gamma_P = 1_P$ and Γ_P is therefore an algebraic decomposition of P.

Simple vertex figures. A vertex figure in $\mathbb{R}^3$ can be fairly complicated, and it is desirable to further decompose it into simpler terms. These terms would be half-spaces, wedges (intersections of 2 half-spaces), and triangular cones (intersections of 3 half-spaces). This is exemplified under the simplifying assumption the vertex belongs to only 3 planes spanned by facets of P. There are 6 types illustrated in figure 4; the corresponding cases in figure 5 are (a), the complement of (a), (e), the complement of (e), (i), and (j). To describe the alge-

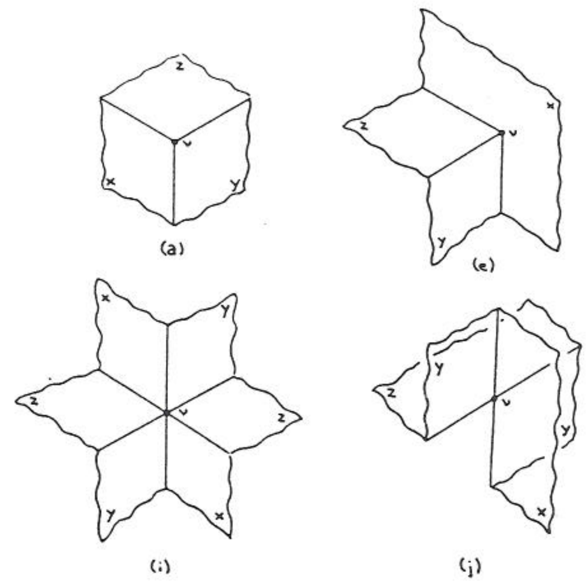

Figure 4: The vertex ν in (a) is either convex or concave, depending on whether the polyhedron lies behind, (a1), or in front, (a2), the surface. Similarly in (e), ν is endpoint of either 1 convex and 2 concave edges, (e1), or of 1 concave and 2 convex edges, (e2). In (i), ν is endpoint of 3 convex and 3 concave edges. In (j), ν is endpoint of 2 convex and 2 concave edges.

braic decomposition in each of the 6 cases, we let 1_x refer to the indicator function of the open half-space in front the plane labeled x. $1_{\bar{x}}$ refers to the indicator function of the open half-space behind this plane. Similar conventions apply to 1_y, $1_{\bar{y}}$, 1_z, and $1_{\bar{z}}$.

(a1) $1_{\text{nff}\,\nu} = 1_x 1_y 1_z$.

(a2) $1_{\text{nff}\,\nu} = 1_{\bar{x}} + 1_{\bar{y}} + 1_{\bar{z}} - 1_{\bar{x}}1_{\bar{y}} - 1_{\bar{x}}1_{\bar{z}} - 1_{\bar{y}}1_{\bar{z}} + 1_{\bar{x}}1_{\bar{y}}1_{\bar{z}}$.

(e1) $1_{\text{nff}\,\nu} = 1_x + 1_y 1_z - 1_x 1_y 1_z$.

(e2) $1_{\text{nff}\,\nu} = 1_{\bar{x}}1_{\bar{y}} + 1_{\bar{x}}1_{\bar{z}} - 1_{\bar{x}}1_{\bar{y}}1_{\bar{z}}$.

(i) $1_{\text{nff}\,\nu} = 1_x 1_y + 1_x 1_z + 1_y 1_z - 2 \cdot 1_x 1_y 1_z$.

(j) $1_{\text{nff}\,\nu} = 1_x 1_{\bar{y}} + 1_x 1_{\bar{z}} + 1_y 1_{\bar{z}} - 1_x 1_{\bar{y}} 1_{\bar{z}} - 1_x 1_y 1_{\bar{z}}$.

For an illustration consider figures 2 and 3, where vertices, edges, and facets are labeled with the coefficients obtained after transforming negative edge and vertex figures. To improve readability, the coefficients $-1, 0, +1$ of edges and vertices are abbreviated to $-, 0, +$. In figure 2, all facets start out with -1, except the front and the back facets labeled 0. The edges have initial coefficients $+1$. The decomposition of edge figures flips the signs of the 4 concave edges and increments the containing facets by 1 for each concave edge. The vertices have initial coefficients -1. There are 8 vertices of type (e2), and the decomposition of their vertex figures flips their sign and decrements the containing convex edges by 1 for each type (e2) vertex. In figure 3, the decomposition of the figures of 12 concave edges and 8 type (a2) vertices have cancelling effects.

4 Concepts in Dimension d

The algebraic decompositions discussed in sections 2 and 3 are special cases of the d-dimensional theorem stated in section 5. This section introduces the necessary concepts and notation, along with results relevant to the theorem and its proof.

Basic definitions. The *affine hull* of S is $\operatorname{aff} S = \{x = \sum_i t_i p_i \mid \sum_i t_i = 1\}$. S is *affinely independent* if $\operatorname{aff} S' \neq \operatorname{aff} S$ for every proper $S' \subset S$. A *k-flat* is the affine hull of an affinely independent collection of $k+1$ points. A *hyperplane* is a $(d-1)$-flat. The *convex hull* of S is $\operatorname{conv} S = \{x = \sum_i t_i p_i \mid \sum_i t_i = 1, t_i \geq 0\}$. A *k-simplex* is the convex hull of an affinely independent collection of $k+1$ points. The notions of affine and convex hulls are extended to infinite sets S by taking sums over finite subsets.

For two points $x, y \in \mathbf{R}^d$, $|xy|$ is the Euclidean distance between them, and $\|x\|$ is the Euclidean distance of x from the origin of $\mathbf{R}^d$. The *unit open ball* is $b = \{x \in \mathbf{R}^d \mid \|y\| < 1\}$. The open ball with radius $\varrho > 0$ and center y is $y + \varrho b$.

The topological concepts of interior and boundary of a set are consistently used in a relative sense. The *interior* of $P \subseteq \mathbf{R}^d$ relative to its affine hull, $\operatorname{aff} P$, is the set $\operatorname{int} P$ of points $y \in P$ for which there is a sufficiently small $\varrho > 0$ with $P \cap (y + \varrho b) = \operatorname{aff} P \cap (y + \varrho b)$. The *boundary* is $\operatorname{bd} P = P - \operatorname{int} P$. P is *open* if $P = \operatorname{int} P$, and it is *closed* if the complement, $\operatorname{aff} P - P$, is open. The *closure* of P is $\operatorname{cl} P = \operatorname{aff} P - \operatorname{int}(\operatorname{aff} P - P)$.

Two sets $M, N \subseteq \mathbf{R}^d$ are *homeomorphic* if there is a continuous bijection $f : M \to N$ with continuous inverse, f^{-1}. Define

$$\mathbf{H}^\ell = \{x = (\xi_1, \xi_2, \ldots, \xi_\ell) \in \mathbf{R}^\ell \mid \xi_1 \geq 0\}.$$

M is an *ℓ-manifold with boundary* if for every $y \in M$ there is a sufficiently small $\varrho > 0$ so $M \cap (y + \varrho b)$ is homeomorphic to either $\mathbf{R}^\ell$ or $\mathbf{H}^\ell$. The points with neighborhood homeomorphic to $\mathbf{H}^\ell$ form the boundary of M.

Convex polyhedra. A *convex polyhedron* in $\mathbf{R}^d$ is the common intersection of finitely many closed half-spaces, $\bigcap H$. Its *dimension* is one less than the maximum number of affinely independent points it contains and is denoted by $\dim \bigcap H$. Its *co-dimension* is $\operatorname{codim} \bigcap H = d - \dim \bigcap H$. If $k = \dim \bigcap H$, we refer to $\bigcap H$ as a *convex k-polyhedron*. A hyperplane, $\operatorname{bd} g$, bounding a closed half-space, g, *supports* $\bigcap H$ if $\bigcap H \subseteq g$ and $\bigcap H \cap \operatorname{bd} g \neq \emptyset$. A *proper face* is the intersection of $\bigcap H$ with a supporting but not containing hyperplane, $\eta = \bigcap H \cap \operatorname{bd} g \neq \bigcap H$. η itself is a convex polyhedron and all faces of η are also faces of $\bigcap H$. We consider $\bigcap H$ as its only *improper face* and write $\Phi_H = \Phi(\bigcap H)$ for the set of all faces, proper and improper.

We are mostly interested in the full-dimensional case, when $\dim \bigcap H = d$. For a face η, define $H_\eta = \{h \in H \mid \eta \subseteq \operatorname{bd} h\}$. The affine hull is $\operatorname{aff} \eta = \bigcap H'_\eta$, where $H'_\eta = \{\operatorname{bd} h \mid h \in H_\eta\}$. The *face figure* of η is $\operatorname{ff} \eta = \bigcap H_\eta$, and the *negative face figure* is $\operatorname{nff} \eta = \bigcap \bar{H}_\eta$, where $\bar{H}_\eta = \{\bar{h} = \mathbf{R}^d - h \mid h \in H_\eta\}$. Note that $\operatorname{nff} \eta$ is the reflection of $\operatorname{int} \operatorname{ff} \eta$ through $\operatorname{aff} \eta$. These definitions also apply to the improper face, $\eta = \bigcap H$. In this case, $H_\eta = \bar{H}_\eta = \emptyset$ and $\operatorname{aff} \eta = \operatorname{ff} \eta = \operatorname{nff} \eta = \mathbf{R}^d$. Recall $1_{\operatorname{nff} \eta}$ is the indicator function of the negative face figure, $\operatorname{nff} \eta$. Using the Euler-Poincaré formula for convex polyhedra one can prove that

$$\Gamma_{\cap H}(x) = \sum_{\eta \in \Phi_H} (-1)^{\operatorname{codim} \eta} \cdot 1_{\operatorname{nff} \eta}(x) \qquad (2)$$

is an algebraic decomposition of $\bigcap H$. This result can also be proved directly, see [7], and we state it for later reference.

PROPOSITION 4.1 Let $\bigcap H \subseteq \mathbf{R}^d$ be a convex d-polyhedron. Then $\Gamma_{\cap H} = 1_{\cap H}$.

Simplicial complexes. An *abstract simplicial complex*, $\mathcal{A}$, is a finite system of sets so $X \in \mathcal{A}$ and $Y \subseteq X$ implies $Y \in \mathcal{A}$. $X \in \mathcal{A}$ is referred to as an *abstract simplex* and its *dimension* is $\dim X = \operatorname{card} X - 1$. The *vertex set* is $\operatorname{Vert} \mathcal{A} = \bigcup \mathcal{A}$. The *Euler characteristic* of $\mathcal{A}$ is

$$\chi(\mathcal{A}) = \sum_{\emptyset \neq X \in \mathcal{A}} (-1)^{\dim X}.$$

$\mathcal{A}$ has a geometric interpretation obtained by mapping elements of $\operatorname{Vert} \mathcal{A}$ to points in some Euclidean space. Specifically, let

$$\varepsilon : \operatorname{Vert} \mathcal{A} \to \mathbf{R}^e$$

so $\text{conv}\,\varepsilon(X) \cap \text{conv}\,\varepsilon(Y) = \text{conv}\,\varepsilon(X \cap Y)$ for all $X, Y \in \mathcal{A}$. Such a map ε always exists provided e is sufficiently large. The $k+1 = \dim X + 1$ points in $\varepsilon(X)$ are necessarily affinely independent, which implies $\text{conv}\,\varepsilon(X)$ is a k-simplex. The resulting set of simplices,

$$\mathcal{K} = \{\text{conv}\,\varepsilon(X) \mid X \in \mathcal{A}\},$$

is a *(geometric) simplicial complex*. The *Euler characteristic* is $\chi(\mathcal{K}) = \chi(\mathcal{A})$, and the *polyhedron* defined by $\mathcal{K}$ is $|\mathcal{K}| = \bigcup \mathcal{K}$.

A most useful property of the Euler characteristic is its invariance over all simplicial complexes, $\mathcal{L}$, defining the same polyhedron, $P = |\mathcal{L}| = |\mathcal{K}|$. Hence, $\chi(P) = \chi(\mathcal{K})$ is well defined. This follows from the stronger result that the alternating sum of simplex numbers equals the alternating sum of betti numbers. This is known as the Euler-Poincaré formula, originally proved in [20, 21], see also [17] or [23]. We state the result restricted to polyhedra for later reference.

PROPOSITION 4.2 Let P be the polyhedron defined by a simplicial complex in $\mathbb{R}^d$. Then

$$\chi(P) = \sum_{i=0}^{d-1} (-1)^i \cdot \beta_i(P).$$

The numbers $\beta_i = \beta_i(P)$ are called the *betti numbers* of P. They can be defined independent of any decomposition of P. However, a full-fledged formal definition is difficult to fit into this paper and not necessary. In short, β_i is the rank of the i-th homology group. Intuitively, β_0 is the number of components, and for $i \geq 1$, β_i is the number of (non-homologous and independent) i-dimensional cycles in P that do not bound.

Covers and nerves. The Euler characteristic of P can also be computed from a system of sets covering P. A *closed convex cover* of P is a finite collection, $\mathcal{C}$, of closed convex sets with $P = \bigcup \mathcal{C}$. For example, every simplicial complex with polyhedron P is such a cover. The *nerve* of $\mathcal{C}$ is the system of subsets with non-empty common intersections,

$$\text{Nrv}\,\mathcal{C} = \{X \subseteq \mathcal{C} \mid \bigcap X \neq \emptyset\}.$$

Note that $\text{Nrv}\,\mathcal{C}$ is an abstract simplicial complex. The nerve lemma, first proved by Leray [13] but see also [2], implies the betti numbers of $\text{Nrv}\,\mathcal{C}$ are the same as the betti numbers of P. Together with the Euler-Poincaré formula, this implies the following result used in the proof of our main theorem.

PROPOSITION 4.3 Let $\mathcal{C}$ be a closed convex cover of a polyhedron P in $\mathbb{R}^d$. Then $\chi(P) = \chi(\text{Nrv}\,\mathcal{C})$.

We will also need the Euler characteristic of the interior of a polyhedron, which is in general not a polyhedron. It is still possible to find a closed convex cover $\mathcal{C}$ of $\text{int}\,P$; the sets in $\mathcal{C}$ are closed with respect to $\text{int}\,P$ albeit not necessarily with respect to $\text{aff}\,P$. We define $\chi(\text{int}\,P) = \chi(\text{Nrv}\,\mathcal{C})$. The nerve lemma implies $\chi(\text{int}\,P)$ is well defined and equal to the alternating sum of betti numbers, just as in the closed case considered in proposition 4.3.

Non-convex polyhedra. In many situations the following definition of a polyhedron is more convenient than the one given above based on simplicial complexes. If restricted to the bounded case the two definitions are equivalent. A *polyhedron* in $\mathbb{R}^d$ is the union of finitely many convex polyhedra,

$$P = \bigcup_{\ell \in \Lambda} \bigcap H_\ell.$$

P is a *k-polyhedron* if $\dim \bigcap H_\ell = k$ for all $\ell \in \Lambda$. We are mostly interested in the full-dimensional case, when $k = d$. We restrict ourselves to bounded polyhedra. While not essential, this restriction is convenient and simplifies some of the arguments.

The concepts of face and face figure are extended from the convex to the non-convex case by considering neighborhoods of points. The $(d-1)$-sphere is $s = \mathbb{S}^{d-1} = \{x \in \mathbb{R}^d \mid \|x\| = 1\}$, and the sphere with radius $\varrho > 0$ and center $y \in \mathbb{R}^d$ is $y + \varrho s$. Below a certain positive threshold for ϱ, all intersections $P \cap (y + \varrho s)$ are homothetic. Assume ϱ is below this threshold, and let $Q_\varrho = (P - y) \cap \varrho s$ be the translation of the intersection to the origin. The *point figure* of y is

$$\text{pf}\,y = y + \bigcup_{t \geq 0} t \cdot Q_\varrho.$$

For example, $\text{pf}\,y = \mathbb{R}^d$ if $y \in \text{int}\,P$, and $\text{pf}\,y = \emptyset$ if $y \notin P$. If P is a convex d-polyhedron and $y \in \text{bd}\,P$ then $\text{pf}\,y$ is the face figure of the face φ with $x \in \text{int}\,\varphi$.

The point figures can be used to classify the points of P into interiors of faces. Specifically, the *face* containing a point y in its interior is

$$\varphi = \text{cl}\,\{x \in \mathbb{R}^d \mid \text{pf}\,x = \text{pf}\,y\},$$

and the *dimension* of φ is $\dim \varphi = \dim \text{aff}\,\varphi$. The *face figure* of φ is $\text{ff}\,\varphi = \text{pf}\,y$, and the *negative face figure* is the reflection of the interior, $\text{nff}\,\varphi = 2y - \text{int}\,\text{pf}\,y$. We say a polyhedron P has the *manifold property* if P is a manifold with boundary and so is each face of P.

5 Main Theorem

Let P be a d-polyhedron with face set $\Phi = \Phi(P)$. As usual, $1_{\mathsf{nff}\,\varphi}$ is the indicator function of the negative face figure of $\varphi \in \Phi$. In analogy to the convex case we define

$$\Gamma_P(x) \;=\; \sum_{\varphi \in \Phi} (-1)^{\operatorname{codim}\varphi} \cdot \chi(\varphi) \cdot 1_{\mathsf{nff}\,\varphi}(x). \quad (3)$$

The only difference to (2) is the Euler characteristic of faces, which naturally enters the formula. The main result of this paper says that with the above generalization of definitions, proposition 4.1 also holds in the non-convex manifold case.

Main Theorem Let P be a d-polyhedron with manifold property in $\mathbb{R}^d$. Then $\Gamma_P = 1_P$.

Proof. The argument consists of five steps. First, P is covered with convex polyhedra. Second, straightforward inclusion-exclusion over the nerve of this cover leads to an algebraic decomposition of P, which is then refined using proposition 4.1. Third, the decomposition is reduced by removing convex polyhedra of dimension less than d. Fourth and fifth, the remaining terms are rearranged leading to cancellations and eventually to the claimed identity. Details are omitted in the conference version of this paper. □

Without manifold property. The proof makes use of the manifold property only in the third step, where cells $\bigcap X$ of dimension $k < d$ are removed. These cells can possibly affect the value of the indicator function only for points $y \in \operatorname{bd} P$. Without the manifold property, the Euler characteristics of a face and its interior are no longer necessarily the same. If we substitute $\chi(\operatorname{int}\varphi)$ for $\chi(\varphi)$ in (3), the formula still correctly indicates membership in P for points $y \in \operatorname{int} P$ and $y \notin P$, even without the manifold property. This will be exploited in section 7. At the expense of a more complicated formula, it is possible to get an indicator function that is correct for all points, also for points on the non-manifold boundary. Details are omitted.

6 Face Figure Decomposition

Except in $d = 2$ dimensions, negative face figures can be arbitrarily complicated, and it is desirable to further decompose them into simpler components. We will discuss the equivalent problem of decomposing (positive) face figures. This section demonstrates an algebraic decomposition into intersections of at most d half-spaces per term. It suffices to consider face figures of vertices.

Simple vertex figures. We begin with the non-degenerate case when ν is *simple*, that is, it belongs to only d hyperplanes spanned by facets of P. The hyperplanes cut $\mathbb{R}^d$ into 2^d (closed) orthants, represented by the vertex set, V, of a d-dimensional cube, $\mathbb{I}^d$. Two

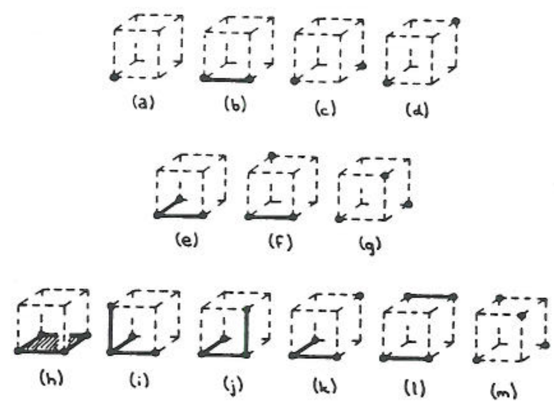

Figure 5: There are twenty cases for choosing a proper subset of the vertices of $\mathbb{I}^3$. The seven cases of picking 1, 2, or 3 vertices have a complementary case each, and the six cases of picking 4 vertices are each symmetric to the complement. The manifold property implies the chosen vertices form a connected induced subcomplex of the cube. This is violated in cases (c), (d), (f), (g), (k), (l), and (m). From the remaining cases drop (b) and (h) because they do not generate a vertex. The cases left are (a), the complement of (a), (e), the complement of (e), (i), and (j), see also figure 4.

orthants are adjacent across a hyperplane iff the corresponding two vertices are connected by an edge. More generally, a k-face of $\mathbb{I}^d$ is a k-dimensional cube and corresponds to 2^k orthants surrounding a common $(d-k)$-flat. The union of these 2^k orthants is the intersection of $d-k$ half-spaces.

The vertex figure of ν is the union of a subset of the orthants, represented by a subset of the vertices, $V_\nu \subseteq V$. For example, if ν is convex then $\mathsf{ff}\,\nu$ is equal to one orthant and $\operatorname{card} V_\nu = 1$. We write V_ν as the union of (not necessarily disjoint) maximal subsets determined by faces of $\mathbb{I}^d$:

$$V_\nu \;=\; F_1 \cup F_2 \cup \ldots \cup F_j.$$

A k-face of $\mathbb{I}^d$ corresponds to a set F_i of cardinality 2^k. It also corresponds to a set H_i of $d-k$ half-spaces so $\bigcap H_i$ is the union of the orthants corresponding to vertices in F_i. The algebraic decomposition of the vertex figure of ν is derived from the nerve of

$F_\nu = \{F_1, F_2, \ldots, F_j\}$:

$$1_{\text{ff}\,\nu}(x) = \sum_{\emptyset \neq X \in \text{Nrv}\, F_\nu} (-1)^{\dim X} \cdot 1_{\text{ff}\,X}(x),$$

where ff X is the intersection of half-spaces that corresponds to $\bigcap X$. More formally, ff $X = \bigcap H_X$, where H_X is the union of all H_i with $F_i \in X$.

Figure 5 illustrates the decomposition by enumerating all different subsets of the 8 vertices of the 3-dimensional cube, $\mathbb{I}^3$. An interesting case is (i) with $F_\nu = \{F_1, F_2, F_3\}$ and $F_1 = \{000, 001\}$, $F_2 = \{000, 010\}$, $F_3 = \{000, 100\}$. The pairwise intersections are the same as the triple intersection, $\bigcap F_\nu = \{000\}$. Hence, $\{000\}$ is counted $-3 + 1 = -2$ times, which explains the factor -2 in the decomposition formula (i) in section 3.

Perturbation. In the general and possibly degenerate case, a vertex ν of P belongs to an arbitrary number of hyperplanes spanned by facets. These hyperplanes decompose $\mathbb{R}^d$ into a finite number of d-dimensional convex polyhedra or *chambers*. Each chamber is a cone with apex at ν. The vertex figure of ν is the union of a subset of these chambers.

We can reduce the general case to many simple cases by perturbing the hyperplanes into non-degenerate position. The perturbed hyperplanes decompose $\mathbb{R}^d$ into a new collection of chambers. Assume the perturbation is sufficiently small so all non-degenerate features are retained. To explain this in more detail we consider the *arrangement*, $A = A(H)$, defined by the set, H, of original hyperplanes. The k-dimensional faces of the chambers are the *k-cells* of A. A k-cell is *simple* if it belongs to exactly $d-k$ hyperplanes in H. All chambers are simple.

Let H' be the set of perturbed hyperplanes defining another arrangement, $A' = A(H')$. All cells of A' are simple. Each cell α of A corresponds to a collection of cells in A'. To see this consider a continuous motion of the hyperplanes in H' to their positions in H. The motion changes A' to A. The cells in A' that *correspond* to $\alpha \in A$ are the ones that belong to a neighborhood of α before they collapse and they do not belong to any small neighborhood of any face of α. To "retain non-degenerate features" means each simple cell of A correspond to a single face of A'. Each original chamber thus corresponds to a unique perturbed chamber.

The *perturbed* vertex figure is the union of a subset of the chambers in A', namely the ones corresponding to the chambers and cells of A contained in ff ν. This union is of course no longer a vertex figure in the strict sense. The original vertex, ν, corresponds to a collection of faces of the perturbed vertex figure. Each face in this collection has a simple face figure, which has a simple algebraic decomposition as described above. The algebraic decomposition of ff ν is the alternating sum of these simple decompositions, each transferred back to the unperturbed state.

Boolean expressions. The second form of the main result is an algebraic decomposition of P into open cones, each the intersection of d or fewer half-spaces:

$$\Gamma_P(x) = \sum_{\ell \in \Lambda} a_\ell \cdot 1_{\cap H_\ell}(x).$$

This form is obtained with or without the method of perturbation, depending on whether there are non-simple k-faces. In any case, $\Gamma_P(x) \in \{0, 1\}$ for all $x \in \mathbb{R}^d$.

Consider taking terms modulo 2. Let $\Lambda' \subseteq \Lambda$ contain all indices with $a_\ell = 1 \pmod 2$. Then

$$\Gamma_P(x) = \sum_{\ell \in \Lambda'} 1_{\cap H_\ell}(x) \pmod 2.$$

In words, $x \in P$ iff it belongs to an odd number of cones $\bigcap H_\ell$, $\ell \in \Lambda'$. This can be expressed by taking symmetric differences of the cones:

$$P = \bigoplus_{\ell \in \Lambda'} \bigcap H_\ell. \qquad (4)$$

Relation (4) is what we call the third form of the main result. If all faces of P are simple, the length of the expression in (4) is at most proportional to the number of faces. The constant of proportionality is less than 3^d. In the non-simple case, the perturbation can significantly increase the number of terms if the polyhedron is highly degenerate.

7 Applications

The main theorem is relevant to various computational and geometric questions about polyhedra. Via limit considerations it leads to statements about non-polyhedral objects. This section briefly discusses some of its applications.

Point classification. A common algorithmic problem in solid modeling is the classification of a query point, x, relative to a given polyhedron, $P \subseteq \mathbb{R}^3$, see e.g. [16, chapter 9]. Assume P has the manifold property. The boundary representation of P is readily translated into the algebraic decomposition given in (3). In case of degeneracies, a perturbation can be simulated

[8] and vertex figures can be further decomposed by local plane arrangement constructions [6]. This results in an algebraic decomposition where each terms is the intersection of 1, 2, or 3 half-spaces. All half-spaces are bounded by facet planes.

For a facet ϕ of P let $h(\phi)$ be the closed half-space locally on the same side of the plane aff ϕ as P. It suffices to decide, for each facet ϕ of P, whether or not $x \in h(\phi)$. If there are n facets we get n bits, 0 for $x \in h(\phi)$ and 1 for $x \notin h(\phi)$. The algebraic decomposition specifies the way they are combined to give the global answer. As explained above, it suffices to work with the parity and to reduce algebraic to boolean operations. Specifically, logical and replaces multiplication and xor replaces addition. Since all operations are independent, the formula lends itself to the construction of a parallel circuit with n input gates, logarithmic depth, and one output gate.

Volume by integration. The d-dimensional volume of P can be computed by integrating Γ_P over a bounded domain, e.g. a d-cube enclosing the polyhedron. Γ_P is a finite sum of simple terms, and the order of summation and integration can be reversed. The result is an inclusion-exclusion formula for the volume:

$$\begin{aligned} \text{vol}\, P &= \int_{\mathbf{I}^d} \Gamma_P(x)dx \\ &= \sum_{\varphi \in \Phi} (-1)^{\text{codim}\,\varphi} \cdot \chi(\varphi) \cdot \int_{\mathbf{I}^d} 1_{\text{nff}\,\varphi}(x)dx. \end{aligned}$$

The same technique can be used to compute the centroid or any other polynomial over P, see [1]. As mentioned at the end of section 5, the algebraic decomposition in (3) gives correct indication for all points $x \notin \text{bd}\, P$, even without the manifold property, provided $\chi(\text{int}\,\varphi)$ is substituted for $\chi(\varphi)$. With this change the above volume formula applies also to non-manifold polyhedra since bd P has measure zero and does not affect the result of the integration.

Spherical balls. A reasonably popular method in computer graphics models 3-dimensional bodies as the union and intersection of spherical balls, see e.g. [22]. Let

$$S = \bigcup_\ell \bigcap B_\ell \qquad (5)$$

be such a body in $\mathbf{R}^3$. We construct a polyhedron $P \subseteq \mathbf{R}^4$ so S is the stereographic projection of $\mathbf{S}^3 \cap P$.

$\mathbf{S}^3$ is the set of points at unit distance from the origin of $\mathbf{R}^4$. Identify $\mathbf{R}^3$ with the hyperplane $\xi_4 = -1$ and define $N = (0,0,0,1)$. *Stereographic projection* is the map

$$\pi : \mathbf{S}^3 - \{N\} \rightarrow \mathbf{R}^3$$

defined so N, $x \in \mathbf{S}^3 - \{N\}$, and $\pi(x)$ are collinear. π is bijective and π^{-1} exists. For each ball $b \subseteq \mathbf{R}^3$ let h_b be the half-space in $\mathbf{R}^4$ so $h_b \cap \mathbf{S}^3 = \pi^{-1}(b)$. Now replace each ball b in (5) by h_b. The result is a polyhedron

$$P = \bigcup_\ell \bigcap H_\ell$$

in $\mathbf{R}^4$. Clearly, $S = \pi(\mathbf{S}^3 \cap P)$ and we can generalize the three forms of the main theorem, the method for point classification, and the formula for volume computation from polyhedra to bodies S. The thus obtained volume formula generalizes the results in [7, 18] for measuring proteins modeled as unions of balls.

Angle formulas. The *angle*, ang φ, at a face φ of P is the fraction of a sufficiently small sphere centered at a point $x \in \text{int}\,\varphi$ that lies inside P. The classical Gram-Sommerville relation for a convex polyhedron is

$$\sum_{\varphi \in \Phi} (-1)^{\text{codim}\,\varphi} \cdot \text{ang}\,\varphi = 0,$$

see e.g. [10]. The relation is sometimes named after Schläfli whose observation preceded the work of Gram and of Sommerville. A generalization to non-convex polyhedra can be found in Chen [3]. For polyhedra P with manifold property, the generalized relation is

$$\sum_{\varphi \in \Phi} (-1)^{\text{codim}\,\varphi} \cdot \chi(\varphi) \cdot \text{ang}\,\varphi = 0, \qquad (6)$$

The similarity of this form with the definition of Γ_P in (3) is not coincidental. (6) can be obtained by integrating Γ_P over the unit sphere, $\mathbf{S}^{d-1}$. If we let P shrink towards the origin, the term $\int_{\mathbf{S}^{d-1}} 1_{\text{nff}\,\varphi}(x)dx$ approaches the angle at φ. The entire integral vanishes because all points of $\mathbf{S}^{d-1}$ lie outside P. Relation (6) is the limit of the integral relation.

Observe the above argument is not affected if P does not have the manifold property. In this case, $\chi(\text{int}\,\varphi)$ is to be substituted for $\chi(\varphi)$, just as in the paragraph about volume. The argument also goes through if the indicator function is multiplied with any reasonable density distribution over $\mathbf{S}^{d-1}$. It follows (6) holds in these more general cases.

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