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TRIANGULATING TOPOLOGICAL SPACES*

HERBERT EDELSBRUNNER
Department of Computer Science, University of Illinois at Urbana-Champaign
Urbana, Illinois 61801, USA.

and

NIMISH R. SHAH
Mentor Graphics, 1001 Ridder Park Drive
San Jose, California 95131, USA.
e-mail: nimish_shah@MENTORG.COM

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ABSTRACT

Given a subspace $\mathbb{X} \subseteq \mathbb{R}^d$ and a finite set $S \subseteq \mathbb{R}^d$, we introduce the Delaunay complex, $\mathcal{D}_{\mathbb{X}}$, restricted by $\mathbb{X}$. Its simplices are spanned by subsets $T \subseteq S$ for which the common intersection of Voronoi cells meets $\mathbb{X}$ in a non-empty set. By the nerve theorem, $\bigcup \mathcal{D}_{\mathbb{X}}$ and $\mathbb{X}$ are homotopy equivalent if all such sets are contractible. This paper proves a sufficient condition for $\bigcup \mathcal{D}_{\mathbb{X}}$ and $\mathbb{X}$ be homeomorphic.

Keywords: Combinatorial topology, geometric modeling, grid generation; topological spaces, manifolds, coverings, nerves, regular complexes, simplicial complexes, triangulations; Voronoi cells, Delaunay complexes; homotopy equivalence, homeomorphisms.

1. Introduction

This paper studies the problem of constructing simplicial complexes that represent or approximate a geometric object in some finite-dimensional Euclidean space, $\mathbb{R}^d$. We refer to the geometric object as a topological space or subspace of $\mathbb{R}^d$. This problem arises in geometric modeling and finite element analysis, and it is a special case of the grid generation problem.[20] It is special because we only consider grids or complexes made up of simplices. The problem can be divided into two questions:

- How do we choose the points or vertices of the grid?

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- How do we connect the vertices using edges, triangles, and higher-dimensional simplices?

In this paper we concentrate on the second question. In particular, given a subspace and a finite point set in $\mathbb{R}^d$, we give an unambiguous rule for constructing a simplicial complex representing the subspace. Topological properties of this simplicial complex, such as whether its domain is homotopy equivalent or homeomorphic to the subspace, can be studied based on local interactions between the subspace and the Voronoi neighborhoods of the points. This leads us back to the first question: additional points can be chosen so they improve the local interaction patterns. How this can be done in a concrete, possibly three-dimensional setting ought to be the subject of future investigations.

1.1. Simplicial Complexes and Triangulations

An affinely independent point set $T \subseteq \mathbb{R}^d$ defines the *simplex* $\sigma_T = \operatorname{conv} T$. Its *dimension* is $k = \dim \sigma_T = \operatorname{card} T - 1$, and it is also referred to as a *k-simplex*. The points of T are the *vertices* of σ_T. A *simplicial complex*, $\mathcal{K}$, is a finite collection of simplices that satisfies the following two properties: if $\sigma_T \in \mathcal{K}$ and $U \subseteq T$ then $\sigma_U \in \mathcal{K}$, and if $\sigma_T, \sigma_V \in \mathcal{K}$ then $\sigma_T \cap \sigma_V = \sigma_{T \cap V}$. The first property implies $\emptyset \in \mathcal{K}$, and the first and second properties combined imply $\sigma_T \cap \sigma_V \in \mathcal{K}$. The *vertex set* of $\mathcal{K}$ is $\operatorname{Vert} \mathcal{K} = \bigcup_{\sigma_T \in \mathcal{K}} T$, the *dimension* is $\dim \mathcal{K} = \max_{\sigma \in \mathcal{K}} \dim \sigma$, and the *underlying space* is $\bigcup \mathcal{K} = \bigcup_{\sigma \in \mathcal{K}} \sigma$. A *subcomplex* of $\mathcal{K}$ is a simplicial complex $\mathcal{L} \subseteq \mathcal{K}$.

A particular simplicial complex defined by a non-degenerate finite set $S \subseteq \mathbb{R}^d$ is the *Delaunay complex*, $\mathcal{D} = \mathcal{D}_S$.[5] It consists of all simplices σ_T, $T \subseteq S$, for which there exists an open ball, B, with $S \cap \operatorname{cl} B = T$ and $S \cap B = \emptyset$. Given S, $\mathcal{D}$ is unique, $\dim \mathcal{D} = \min\{d, \operatorname{card} S - 1\}$, and $\bigcup \mathcal{D} = \operatorname{conv} S$. In computational geometry, $\mathcal{D}$ is usually referred to as the Delaunay 'triangulation' of S.[6,17] To avoid confusion with the topology notion of a triangulation, which is adopted in this paper, we choose to call $\mathcal{D}$ a complex. Following the tradition in combinatorial topology, a *triangulation* of a topological space $\mathbb{X}$ is a simplicial complex $\mathcal{K}$ together with a homeomorphism between $\mathbb{X}$ and $\bigcup \mathcal{K}$.[16,18] If there exists a simplicial complex $\mathcal{K}$ such that $\bigcup \mathcal{K}$ is homeomorphic to $\mathbb{X}$, then $\mathbb{X}$ is *triangulable*.

1.2. Outline

Our approach to constructing a simplicial complex that represents a topological space $\mathbb{X} \subseteq \mathbb{R}^d$ is based on a finite set $S \subseteq \mathbb{R}^d$ and the Delaunay complex of this set, $\mathcal{D} = \mathcal{D}_S$. Section 2 introduces the concept of a Delaunay complex $\mathcal{D}_{\mathbb{X}}$ restricted by $\mathbb{X}$, which is a subcomplex of $\mathcal{D}$. This concept is a common generalization of ideas developed by Martinetz and Schulten,[14] Chew,[3] and Edelsbrunner.[7] $\mathcal{D}_{\mathbb{X}}$ is defined for every $\mathbb{X} \subseteq \mathbb{R}^d$ and every non-degenerate finite $S \subseteq \mathbb{R}^d$. If S satisfies the assumptions of the nerve theorem, see section 2, then $\bigcup \mathcal{D}_{\mathbb{X}}$ and $\mathbb{X}$ are homotopy equivalent. Section 3 presents some topological concepts and discusses the meaning of non-degeneracy in detail. Sections 4 and 5 study conditions on S that guarantee

$\bigcup \mathcal{D}_{\mathbb{X}}$ be homeomorphic to $\mathbb{X}$. An explicit construction of a homeomorphism is also given. Section 6 mentions directions for further research.

2. Restricted Delaunay Complexes

2.1. Coverings and Nerves

Let $\mathbb{X} \subseteq \mathbb{R}^d$ be a topological space and $S \subseteq \mathbb{R}^d$ a finite point set. We assume non-degenerate position of the points in S, which generically means that anything vanishing under a slight perturbation of the points is precluded. For example, we require that $T \subseteq S$ be affinely independent if $\operatorname{card} T \leq d + 1$, and that no $d + 2$ points of S be cospherical. Further particulars of this assumption will be discussed in section 3. The *Voronoi cell* of $p \in S$ is

$$V_p = \{x \in \mathbb{R}^d \mid |px| \leq |qx|,\ q \in S\},$$

where $|yz|$ denotes the Euclidean distance between points $y, z \in \mathbb{R}^d$. The collection of Voronoi cells is $\mathsf{V} = \mathsf{V}_S = \{V_p \mid p \in S\}$.[21] The *Voronoi cell restricted to* $\mathbb{X}$ of $p \in S$ is $V_{p,\mathbb{X}} = \mathbb{X} \cap V_p$, and the collection of restricted Voronoi cells is $\mathsf{V}_{\mathbb{X}} = \mathsf{V}_{S,\mathbb{X}} = \{V_{p,\mathbb{X}} \neq \emptyset \mid p \in S\}$. For a subset $T \subseteq S$ we have corresponding subsets $\mathsf{V}_T = \{V_p \mid p \in T\} \subseteq \mathsf{V}$ and $\mathsf{V}_{T,\mathbb{X}} = \{V_{p,\mathbb{X}} \neq \emptyset \mid p \in T\} \subseteq \mathsf{V}_{\mathbb{X}}$. We will consider their common intersections, $\bigcap \mathsf{V}_T$ and $\bigcap \mathsf{V}_{T,\mathbb{X}} = \mathbb{X} \cap \bigcap \mathsf{V}_T$. In the computational geometry literature common intersections of Voronoi cells are usually referred to as Voronoi vertices, Voronoi edges, and higher-dimensional Voronoi faces.[6,17]

A *covering* of $\mathbb{X}$ is a collection C of subsets of $\mathbb{X}$ so that $\mathbb{X} = \bigcup \mathsf{C}$. It is a *closed (open)* covering if each set in C is closed (open), and it is a *finite* covering if C is finite. For a subset $\mathsf{D} \subseteq \mathsf{C}$ consider the common intersection, $\bigcap \mathsf{D}$. The *nerve* of a finite covering C is

$$\operatorname{Nrv} \mathsf{C} = \{\mathsf{D} \subseteq \mathsf{C} \mid \bigcap \mathsf{D} \neq \emptyset\}.$$

We remark that the nerve can be defined for a finite covering of any abstract set, not just for subsets of $\mathbb{R}^d$. A *geometric realization* of $\operatorname{Nrv} \mathsf{C}$ is a simplicial complex, $\mathcal{K}$, together with a bijection β between C and the vertex set of $\mathcal{K}$, so that $\mathsf{D} \in \operatorname{Nrv} \mathsf{C}$ iff $\beta(\mathsf{D})$ spans a simplex in $\mathcal{K}$.

Observe that the collection of Voronoi cells restricted to $\mathbb{X}$, the set $\mathsf{V}_{\mathbb{X}}$, is a finite closed covering of $\mathbb{X}$. The *Delaunay complex restricted* by $\mathbb{X}$, $\mathcal{D}_{\mathbb{X}} = \mathcal{D}_{S,\mathbb{X}}$, is the geometric realization of $\operatorname{Nrv} \mathsf{V}_{\mathbb{X}}$ defined by $\beta(V_{p,\mathbb{X}}) = p$, for all $p \in S$. That is, $\mathcal{D}_{\mathbb{X}} = \{\sigma_T \mid T \subseteq S,\ \bigcap \mathsf{V}_{T,\mathbb{X}} \neq \emptyset\}$. Following the terminology in the computational geometry literature, we could call $\mathcal{D}_{\mathbb{X}}$ the Delaunay dual of the Voronoi diagram restricted to $\mathbb{X}$: $\bigcap \mathsf{V}_{T,\mathbb{X}} \neq \emptyset$ iff $\sigma_T \in \mathcal{D}$. Note that $\mathcal{D}_{\mathbb{X}}$ is a subcomplex of the Delaunay complex $\mathcal{D} = \mathcal{D}_{\mathbb{R}^d}$ of S. See figure 1 for an example. The nerve theorem of combinatorial topology[1,12,23] sheds some light on the relationship between $\mathbb{X}$, $\mathsf{V}_{\mathbb{X}}$, and $\mathcal{D}_{\mathbb{X}}$. See section 3 for a formal definition of homotopy equivalence and contractibility.

Theorem 1 (nerve) *Let* C *be a finite closed covering of a triangulable space* $\mathbb{X} \subseteq \mathbb{R}^d$ *so that for every* $\mathsf{D} \subseteq \mathsf{C}$, $\bigcap \mathsf{D}$ *is either empty or contractible. Let* $\mathcal{K}$ *be a geometric*

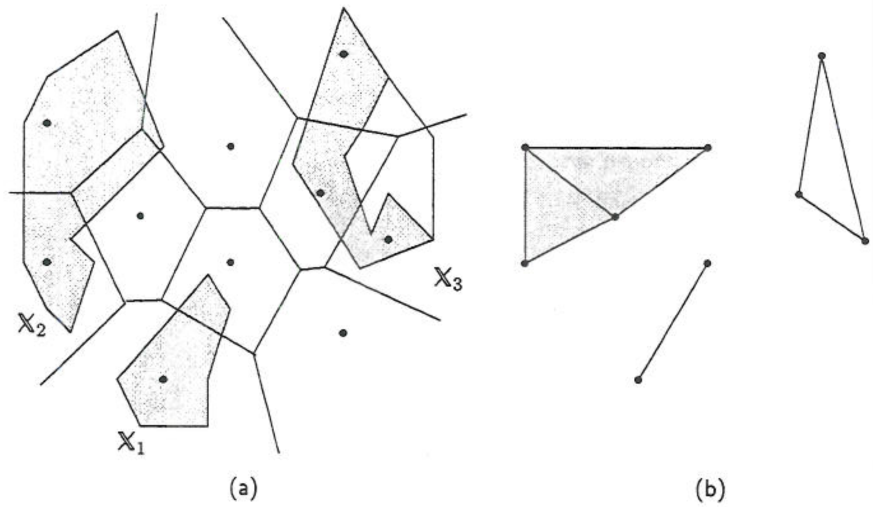

Fig. 1. (a) The Voronoi cells of ten points decompose three spaces, $\mathbb{X}_1$, $\mathbb{X}_2$, and $\mathbb{X}_3$. (b) The three corresponding Delaunay complexes consist of an edge, two triangles sharing an edge, and a cycle of three edges, respectively.

realization of Nrv C. *Then* $\mathbb{X}$ *and* $\bigcup\mathcal{K}$ *are homotopy equivalent.*

In particular, if $\bigcap \mathsf{D}$ is empty or contractible for every $\mathsf{D} \subseteq \mathsf{V}_{\mathbb{X}}$ then $\mathbb{X}$ and $\bigcup\mathcal{D}_{\mathbb{X}}$ are homotopy equivalent.

2.2. Related Earlier Work

Martinetz and Schulten[14] study neural nets modeling a topological space $\mathbb{X}$. In geometric language, a *neural net* consists of finitely many points and edges between them. The algorithm by Martinetz and Schulten constructs the net by choosing a finite set $S \subseteq \mathbb{X}$ and selecting edges based on points sampled from $\mathbb{X}$ found in Voronoi cells associated with point pairs. A cell is associated with every pair $\{p, q\} \subseteq S$ for which $V_p \cap V_q \neq \emptyset$.[6,17] This cell can be interpreted as a "thickened" version of the face common to V_p and V_q, and a point of $\mathbb{X}$ sampled in this cell is taken as evidence that $\mathbb{X} \cap V_p \cap V_q \neq \emptyset$. Martinetz and Schulten call the resulting net the *induced* Delaunay triangulation of S and $\mathbb{X}$; in the limit it is the same as the edge-skeleton of $\mathcal{D}_{\mathbb{X}}$.

Chew[3] introduces a method for constructing a simplicial complex approximating a (two-dimensional) surface in $\mathbb{R}^3$. Let $\mathbb{X}$ be such a surface and $S \subseteq \mathbb{X}$ a finite point set. Three points $p, q, r \in S$ span a triangle in the approximating complex if they lie on the boundary of an open ball $B \subseteq \mathbb{R}^3$ with center in $\mathbb{X}$ so that $S \cap B = \emptyset$. Assuming non-degeneracy, such a ball B exists iff $\mathbb{X} \cap V_p \cap V_q \cap V_r \neq \emptyset$. Chew's method can thus be seen as a special case of our definition of restricted Delaunay complexes.

Finally, Edelsbrunner[7] defines the dual complex of a union of balls in $\mathbb{R}^d$. Consider the case where all balls are equally large. Let $S \subseteq \mathbb{R}^d$ be a finite point set, and define $\mathbb{X} = \{x \in \mathbb{R}^d \mid \min_{p \in S} |xp| \leq \rho\}$, for some fixed positive $\rho \in \mathbb{R}$. The Voronoi cells of S decompose $\mathbb{X}$ into closed convex regions, and the dual complex is defined as the nerve of these regions, geometrically realized by the map $\beta(V_{p,\mathbb{X}}) = p$, for all $p \in S$. We see that it is the same as the Delaunay complex, $\mathcal{D}_{\mathbb{X}}$, restricted by $\mathbb{X}$. The common intersection of any subset of these regions is convex and therefore contractible, so the nerve theorem implies that the underlying space of the dual complex is homotopy equivalent to $\mathbb{X}$.

3. Some Topological Concepts

3.1. Neighborhoods, Homotopies, Homeomorphisms, and Manifolds

An *open ball* in $\mathbb{R}^d$ is a set $B = B(x, \rho) = \{y \in \mathbb{R}^d \mid |yx| < \rho\}$ for some point $x \in \mathbb{R}^d$ and some positive $\rho \in \mathbb{R}$; x is the *center* and ρ the *radius* of B. For $\mathbb{Y} \subseteq \mathbb{X}$, a *neighborhood* of $\mathbb{Y}$ in $\mathbb{X}$ is an open subset of $\mathbb{X}$ that contains $\mathbb{Y}$.

Let $\mathbb{X}$ and $\mathbb{Y}$ be two topological spaces. Two maps $f, g : \mathbb{X} \to \mathbb{Y}$ are *homotopic* if there is a continuous map $h : \mathbb{X} \times [0,1] \to \mathbb{Y}$ with $h(x,0) = f(x)$ and $h(x,1) = g(x)$ for all $x \in \mathbb{X}$. The two spaces, $\mathbb{X}$ and $\mathbb{Y}$, are *homotopy equivalent* if there are continuous maps $f : \mathbb{X} \to \mathbb{Y}$ and $g : \mathbb{Y} \to \mathbb{X}$ so that $g \circ f$ is homotopic to the identity map in $\mathbb{X}$ and $f \circ g$ is homotopic to the identity map in $\mathbb{Y}$. $\mathbb{X}$ is *contractible* if it is homotopy equivalent to a point. For example, the open ball $B(x, \rho)$ is contractible but the unit circle is not contractible.

Topological spaces $\mathbb{X}$ and $\mathbb{Y}$ are *homeomorphic*, written $\mathbb{X} \approx \mathbb{Y}$, if there is a bijective map $\varphi : \mathbb{X} \to \mathbb{Y}$ so that φ and φ^{-1} are continuous. φ is a *homeomorphism* between $\mathbb{X}$ and $\mathbb{Y}$, and $\mathbb{X}$ and $\mathbb{Y}$ are *homeomorphs* of each other. Homeomorphs of open, half-open, and closed balls of various dimensions play an important role in the forthcoming discussions. For $k \geq 0$, let o be the origin of $\mathbb{R}^k$ and define

$$\begin{aligned} \mathbb{H}^k &= \{x = (\xi_1, \ldots, \xi_k) \in \mathbb{R}^k \mid \xi_k \geq 0\}, \\ \mathbb{B}^k &= \{x \in \mathbb{R}^k \mid |x\mathsf{o}| \leq 1\}, \text{ and} \\ \mathbb{S}^{k-1} &= \{x \in \mathbb{R}^k \mid |x\mathsf{o}| = 1\}. \end{aligned}$$

For convenience, we define $\mathbb{R}^k = \mathbb{H}^k = \mathbb{B}^k = \mathbb{S}^k = \emptyset$ if $k < 0$. An *open k-ball* is a homeomorph of $\mathbb{R}^k$, a *half-open k-ball* is a homeomorph of $\mathbb{H}^k$, a *closed k-ball* is a homeomorph of $\mathbb{B}^k$, and a *$(k-1)$-sphere* is a homeomorph of $\mathbb{S}^{k-1}$. For $k \geq 1$ these are disjoint classes of spaces, that is, open balls, half-open balls, closed balls, and spheres are pairwise non-homeomorphic. This is not true for $k = 0$: open, half-open, and closed 0-balls are points, and a 0-sphere is a pair of points.

Our first theorem is about topological spaces that are manifolds, with or without boundary. $\mathbb{X} \subseteq \mathbb{R}^d$ is a *k-manifold without boundary* if each $x \in \mathbb{X}$ has an open k-ball as a neighborhood in $\mathbb{X}$. $\mathbb{X} \subseteq \mathbb{R}^d$ is a *k-manifold with boundary* if each $x \in \mathbb{X}$ has an open or half-open k-ball as a neighborhood in $\mathbb{X}$, and there is at least one $x \in \mathbb{X}$ that has no open k-ball as a neighborhood. The set of points without open k-ball neighborhood forms the *boundary*, $\operatorname{bd} \mathbb{X}$, of $\mathbb{X}$. Note that the boundary of a half-open k-ball is an open $(k-1)$-ball, which is therefore without boundary. From this it follows that the boundary of a k-manifold with boundary is a $(k-1)$-manifold without boundary. The *interior* of a manifold $\mathbb{X}$ is $\text{int}\,\mathbb{X} = \mathbb{X} - \text{bd}\,\mathbb{X}$; it is the set of points with open k-ball neighborhoods. Note that our definition distinguishes between manifolds with and without boundary, which is somewhat non-standard as the set of manifolds without boundary is usually considered a subset of the set of manifolds with boundary. A manifold $\mathbb{X}$ is *compact* if every open covering of $\mathbb{X}$ has a finite subcovering, or equivalently, if it is closed and bounded. A manifold $\mathbb{Y} \subseteq \mathbb{X}$ is a *submanifold* of $\mathbb{X}$.

3.2. Non-degeneracy

In section 4, we are interested in the intersection between manifolds and affine flats. An ℓ-manifold $F \subseteq \mathbb{R}^d$ is an *ℓ-flat* if it is the affine hull of $\ell + 1$ points, or equivalently, it is the intersection of $d - \ell$ hyperplanes, or linear functionals. Let $\mathbb{X} \subseteq \mathbb{R}^d$ be an m-manifold. Intuitively, a point $x \in \mathbb{R}^d$ has d degrees of freedom, and it loses $d-m$ of them if it is constrained to lie in $\mathbb{X}$. Similarly, x looses $d-\ell$ degrees if it is constrained to lie in F. So if $x \in \mathbb{X} \cap F$ then x should have lost $2d - m - \ell$ degrees of freedom. Hence, $\mathbb{X} \cap F$ should be empty if $d-(2d-m-\ell) = m+\ell-d < 0$. In general, we expect x to have $m + \ell - d$ degrees of freedom and $\mathbb{X} \cap F$ to be an $(m + \ell - d)$-manifold. Algorithmically, such a non-degeneracy assumption can be simulated by conceptual perturbation techniques.[8,24]

This intuitive argument can be formalized for smooth or piecewise smooth manifolds. We need some definitions from differential topology. For an open set $\mathbb{X} \subseteq \mathbb{R}^m$, a map $f : \mathbb{X} \to \mathbb{R}^n$ is *smooth* if it has continuous partial derivatives of all orders and at all points of $\mathbb{X}$. For arbitrary $\mathbb{X} \subseteq \mathbb{R}^m$, f is *smooth* if for all $x \in \mathbb{X}$ there exists an open ball $B = B(x, \varepsilon) \subseteq \mathbb{R}^m$ and a smooth map $g : B \to \mathbb{R}^n$ so that g equals f on $\mathbb{X} \cap B$. Topological spaces $\mathbb{X} \subseteq \mathbb{R}^m$ and $\mathbb{Y} \subseteq \mathbb{R}^n$ are *diffeomorphic* if there exists a homeomorphism $\varphi : \mathbb{X} \to \mathbb{Y}$ so that φ and φ^{-1} are smooth. An m-manifold $\mathbb{X}$ with or without boundary is smooth if each $x \in \mathbb{X}$ has a neighborhood diffeomorphic to $\mathbb{R}^k$ or $\mathbb{H}^k$.

Now, let $\mathbb{X}$ be a smooth m-manifold, and let $f : \mathbb{X} \to \mathbb{R}$ be a smooth map. Then $y \in \mathbb{R}$ is a *regular value* of f if for every $x \in f^{-1}(y)$ some partial derivative of f at x is non-zero; otherwise, y is a *critical value* of f. By the preimage theorem in differential topology, the preimage of any regular value is a smooth submanifold of $\mathbb{X}$ with dimension $m-1$, and by Sard's theorem, the set of critical values has measure 0 in $\mathbb{R}$.[9,10] This implies that with probability 1, the intersection between a smooth m-manifold $\mathbb{X}$ and a hyperplane with prescribed normal direction is a smooth $(m-1)$-manifold. Hence, with probability 1, the intersection between $\mathbb{X}$ and an ℓ-flat F with prescribed normal $(d-\ell)$-flat is a smooth $(m+\ell-d)$-manifold. By non-degenerate position of F we mean that this is indeed the case, and it is reasonable to assume non-degenerate position because F just needs to avoid a measure zero set in $\mathbb{R}^{d-\ell}$.

One of the conditions necessary for our results is the non-degeneracy of the intersections between Voronoi cells and the manifold. We thus extend the above notions to Voronoi cells and their intersections. An intersection of Voronoi cells is the common intersection of finitely many closed half-spaces, and thus a *convex polyhedron*. Let P be a convex polyhedron and let $\mathbb{X}$ be a manifold without boundary.

We say that P *intersects* $\mathbb{X}$ *generically* if $\mathbb{X} \cap \mathsf{P} = \emptyset$ or $\mathbb{X} \cap \mathsf{P}$ has the right dimension and $\mathbb{X} \cap \operatorname{int} \mathsf{P} = \operatorname{int}(\mathbb{X} \cap \mathsf{P})$. If $\mathbb{X}$ is a manifold with boundary, then P *intersects* $\mathbb{X}$ *generically* if P intersects $\operatorname{int} \mathbb{X}$ and $\operatorname{bd} \mathbb{X}$ generically. By non-degenerate position of P we mean that P intersects $\mathbb{X}$ generically. Again, this is a reasonable assumption to make.

4. Triangulating Compact Manifolds

We are now ready to state conditions under which the restricted Delaunay complex is a triangulation of $\mathbb{X}$. These conditions will be applied only when $\mathbb{X}$ is a compact manifold. To avoid any misconception, we note that our results do not settle the open question whether or not all compact manifolds are triangulable.

4.1. The Closed Ball Property

Throughout this section we assume non-degenerate position of flats and discrete point sets. Let $m > 0$ and let $\mathbb{X} \subseteq \mathbb{R}^d$ be a compact m-manifold with or without boundary. Let $S \subseteq \mathbb{R}^d$ be a finite point set. We say that S has the *generic intersection property* for $\mathbb{X}$ if for every subset $T \subseteq S$, $\bigcap \mathsf{V}_T$ intersects $\mathbb{X}$ generically. We say that S has the *closed ball property* for $\mathbb{X}$ if for every $\ell \leq m$ and every subset $T \subseteq S$ with $\operatorname{card} T = m + 1 - \ell$, the following two conditions hold:

(B1) $\bigcap \mathsf{V}_{T,\mathbb{X}}$ is either empty or a closed ℓ-ball, and

(B2) $\bigcap \mathsf{V}_{T,\operatorname{bd}\mathbb{X}}$ is either empty or a closed $(\ell - 1)$-ball.

In section 3, we argued that it is reasonable to expect $\mathbb{X} \cap \bigcap \mathsf{V}_T$ be an ℓ-manifold, if $\mathbb{X}$ is a smooth m-manifold. This is reflected in condition (B1). A similar non-degenerate position assumption is implied by (B2) for the boundary of $\mathbb{X}$. We will see that the closed ball property guarantees that dimension is preserved. As an example consider $\mathbb{X}_1$ and $\mathcal{D}_{\mathbb{X}_1}$ in figure 1. Condition (B2) is violated by the common edge of the two Voronoi cells intersecting $\mathbb{X}_1$. Indeed, $\bigcup \mathcal{D}_{\mathbb{X}_1} \not\approx \mathbb{X}_1$ because the dimension of $\mathbb{X}_1$ is two and that of $\bigcup \mathcal{D}_{\mathbb{X}_1}$ is one.

4.2. Two Technical Lemmas

Before proving the first theorem we establish two facts about the closed ball property. The first says that the closed ball property preserves dimension locally, and the second is a statement about the way Voronoi cells intersect a compact manifold and its boundary. We work with arbitrary, that is, possibly non-smooth m-manifolds since we do not need any smoothness properties for our proofs. We do, however, need the non-degenerate intersection property whose introduction was motivated by smooth manifolds. A simplex $\sigma_T \in \mathcal{D}_{\mathbb{X}}$ is a *principal simplex* if there is no proper superset $U \supset T$ with $\sigma_U \in \mathcal{D}_{\mathbb{X}}$. The first fact is formalized in the following lemma.

Lemma 1 (preservation of dimension) *Let $m \geq 1$, let $\mathbb{X} \subseteq \mathbb{R}^d$ be a compact m-manifold with or without boundary, and let $S \subseteq \mathbb{R}^d$ be a non-degenerate finite set* *of points that has the generic intersection property for* $\mathbb{X}$. *If* S *has the closed ball property for* $\mathbb{X}$ *then every principal simplex of* $\mathcal{D}_{\mathbb{X}}$ *is an* m*-simplex.*

Proof. Let $\sigma_T \in \mathcal{D}_{\mathbb{X}}$ be a principal simplex, and define $F' = \bigcap \mathsf{V}_T$ and $F = \bigcap \mathsf{V}_{T,\mathbb{X}}$. By condition (B1), $F = \mathbb{X} \cap F'$ is a closed ℓ-ball, with $\ell = m + 1 - \operatorname{card} T$. Since $\sigma_T \in \mathcal{D}_{\mathbb{X}}$, we have $F \neq \emptyset$, which implies $\ell \geq 0$ and therefore $\operatorname{card} T \leq m+1$. If $\operatorname{card} T = m+1$ then σ_T is an m-simplex and we are done. So suppose $\operatorname{card} T < m+1$. Since σ_T is a principal simplex, it follows that $F \subseteq \operatorname{int} F'$, for otherwise there is a proper face $G' = \bigcap \mathsf{V}_U$ of F' with $\mathbb{X} \cap G' \neq \emptyset$. It follows that $T \subset U$ and $\sigma_U \in \mathcal{D}_{\mathbb{X}}$, which contradicts the principality of σ_T.

Finally, we show that $F \subseteq \operatorname{int} F'$ also leads to a contradiction. As mentioned above, F is a closed ℓ-ball and hence $\operatorname{bd} F$ is an $(\ell-1)$-sphere. This $(\ell-1)$-sphere is contained in $\operatorname{int} F'$, so it must lie in $\operatorname{bd} \mathbb{X}$. Indeed, $\operatorname{bd} F = \operatorname{bd} \mathbb{X} \cap F'$. This contradicts condition (B2), which requires $\operatorname{bd} \mathbb{X} \cap F'$ be a closed $(\ell-1)$-ball. □

For the next lemma we need a classic result on subdivisions of a certain type of complex. A closed ball is called a *cell*, or a k*-cell* if its dimension is k. A finite collection of non-empty cells, $\mathcal{R}$, is a *regular complex* if the cells have pairwise disjoint interiors, and the boundary of each cell is the union of other cells in $\mathcal{R}$. A regular complex generalizes the concept of a simplicial complex by substituting cells for simplices. A subset of $\mathcal{R}$ is a *chain* if its elements can be ordered so that each contains its predecessors and is contained in its successors. Let $\mathsf{C}_{\mathcal{R}}$ be the set of chains in $\mathcal{R}$ and note that $\operatorname{Nrv} \mathsf{C}_{\mathcal{R}}$ is well defined. The result mentioned is that any geometric realization of $\operatorname{Nrv} \mathsf{C}_{\mathcal{R}}$ is homeomorphic to $\bigcup \mathcal{R}$. It can be found in Ref. [4] and Ref. [13] and also in Ref. [2] where it is applied to manifolds subdivided by the cells of a regular complex. Another classic result needed is the weak Schönflies theorem, see e.g. Ref. [19], which implies that if A is a piecewise linear k-sphere and $B \subset A$ is a piecewise linear closed k-ball then $A - \operatorname{int} B$ is also a closed k-ball.

Lemma 2 (complementary closed ball property) *Let* m, $\mathbb{X}$, *and* S *be as in lemma 1. Let* $T \subseteq S$ *be so that* $G = \bigcap \mathsf{V}_{T,\operatorname{bd}\mathbb{X}} \neq \emptyset$, *and define* $F = \bigcap \mathsf{V}_{T,\mathbb{X}}$ *and* $\ell = m - \operatorname{card} T$. *If* S *has the closed ball property for* $\mathbb{X}$ *then* $\operatorname{bd} F - \operatorname{int} G$ *is a closed* ℓ*-ball.*

Proof. By condition (B1), F is a closed $(\ell+1)$-ball, and thus an $(\ell+1)$-manifold with boundary. Let $\mathcal{R}$ consist of all sets $\bigcap \mathsf{V}_{U,\mathbb{X}}$ and $\bigcap \mathsf{V}_{U,\operatorname{bd}\mathbb{X}}$ over all $U \subseteq S$ with $T \subseteq U$. Assuming S has the closed ball property for $\mathbb{X}$, these sets are cells, so $\mathcal{R}$ is a regular complex and $F = \bigcup \mathcal{R}$. Let $\mathcal{K}$ be a geometric realization of $\operatorname{Nrv} \mathsf{C}_{\mathcal{R}}$ and let $\varphi : F \to \bigcup \mathcal{K}$ be a homeomorphism; it exists because of the homeomorphism result mentioned above. Note that $\varphi(\operatorname{bd} F) = \operatorname{bd} \bigcup \mathcal{K}$ is the underlying space of a subcomplex of $\mathcal{K}$, and similarly, $\varphi(G) = \bigcup \mathcal{L}$ for a subcomplex $\mathcal{L} \subseteq \mathcal{K}$. By construction, $\operatorname{bd} \bigcup \mathcal{K}$ is a piecewise linear ℓ-sphere and $\bigcup \mathcal{L} \subset \operatorname{bd} \bigcup \mathcal{K}$ is a piecewise linear closed ℓ-ball. The weak Schönflies theorem implies that $\operatorname{bd} \bigcup \mathcal{K} - \operatorname{int} \bigcup \mathcal{L}$ is a closed ℓ-ball. Since φ is a homeomorphism, $\varphi^{-1}(\operatorname{bd} \bigcup \mathcal{K} - \operatorname{int} \bigcup \mathcal{L}) = \operatorname{bd} F - \operatorname{int} G$ is also a closed ℓ-ball, as claimed. □

4.3. Theorem for Manifolds

We need a few additional definitions. The *barycentric coordinates* of a point x with respect to a simplex σ_T, $T = \{v_0, \ldots, v_k\}$, are real numbers $\xi_0, \ldots, \xi_k$ so that $\sum_{i=0}^{k} \xi_i = 1$ and $\sum_{i=0}^{k} \xi_i v_i = x$; they are unique and non-negative if $x \in \sigma_T$. Let $\mathcal{K}$ and $\mathcal{L}$ be two simplicial complexes, and let $f : \mathrm{Vert}\,\mathcal{K} \to \mathrm{Vert}\,\mathcal{L}$ take the vertices of any simplex in $\mathcal{K}$ to the vertices of a simplex in $\mathcal{L}$. The *simplicial map* implied by f is $g : \bigcup \mathcal{K} \to \bigcup \mathcal{L}$, which maps a point $x \in \sigma_T$, $T = \{v_0, \ldots, v_k\}$, to $g(x) = \sum_{i=0}^{k} \xi_i f(v_i)$. We will use the fact that if f is a bijection then g is a homeomorphism. The *barycenter* of σ_T is $\mathsf{b}_T = \sum_{i=0}^{k} \frac{v_i}{k+1}$, and the *barycentric subdivision* of $\mathcal{K}$ is

$$\mathrm{Sd}\,\mathcal{K} = \{\mathrm{conv}\,\{\mathsf{b}_T \mid \sigma_T \in C\} \mid C \in \mathsf{C}_{\mathcal{K}}\}.$$

Figure 2 (c) shows the barycentric subdivision of the simplicial complex in 2 (b). Note that $\mathrm{Sd}\,\mathcal{K}$ can be constructed inductively by connecting b_T to all simplices

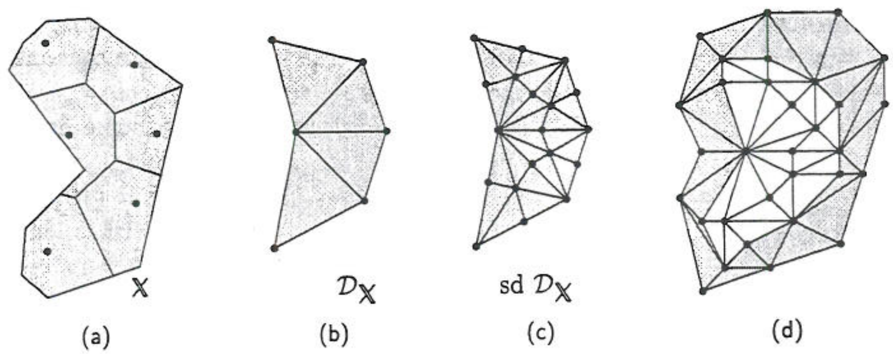

Fig. 2. (a) The space $\mathbb{X}$ is a closed 2-ball in the form of a boomerang. It is covered by the restricted Voronoi cells of six points, which define a regular complex $\mathcal{R}$ with $\bigcup \mathcal{R} = \mathbb{X}$. (b) The restricted Delaunay complex, $\mathcal{D}_{\mathbb{X}}$, consists of four triangles connecting six points. (c) The barycentric subdivision, $\mathrm{Sd}\,\mathcal{D}_{\mathbb{X}}$, has the same underlying space as $\mathcal{D}_{\mathbb{X}}$. (d) A geometric realization of $\mathrm{Nrv}\,\mathsf{C}_{\mathcal{R}}$ consists of a simplicially homeomorphic copy of $\mathrm{Sd}\,\mathcal{D}_{\mathbb{X}}$ (not shaded), surrounded by a layer of triangles (shaded).

subdividing the proper faces of σ_T. The *star* of a vertex $v \in \mathcal{K}$ is $\mathrm{St}\,v = \{\sigma \in \mathcal{K} \mid v \in \sigma\}$.

We will use these tools to show our first result stated in theorem 2 below. The proof constructs a homeomorphism between $\mathbb{X}$ and $\bigcup \mathcal{D}_{\mathbb{X}}$ one step at a time. In this process, the pasting lemma of point set topology[15,18] is employed. It can be stated as follows. If $f : A \to Y$ and $g : B \to Y$ are continuous maps that agree on $A \cap B$ and A, B are closed in $A \cup B$, then $h : A \cup B \to Y$, which agrees with f on A and with g on B is continuous.

Theorem 2 *Let $\mathbb{X} \subseteq \mathbb{R}^d$ be a compact manifold, with or without boundary, and let $S \subseteq \mathbb{R}^d$ be a non-degenerate finite point set that has the generic intersection property for $\mathbb{X}$. If S has the closed ball property for $\mathbb{X}$ then $\bigcup \mathcal{D}_{\mathbb{X}} \approx \mathbb{X}$.*

Proof. For each i, define $\mathsf{V}_i = \{\bigcap \mathsf{V}_{T,\mathbb{X}} \neq \emptyset \mid \mathrm{card}\,T = m + 1 - i\}$ and note that because of the closed ball property all elements in V_i are closed i-balls. We inductively construct simplicial complexes $\mathcal{K}_i$ and homeomorphisms $\varphi_i : \bigcup \mathsf{V}_i \to \bigcup \mathcal{K}_i$, so that $\mathcal{K}_{i-1} \subseteq \mathcal{K}_i$ and φ_i agrees with φ_{i-1} on $\bigcup \mathsf{V}_{i-1}$. When we arrive at $\mathcal{K} = \mathcal{K}_m$ and $\varphi = \varphi_m$ we show there is a simplicial homeomorphism between $\mathcal{K}$ and $\operatorname{Sd} \mathcal{D}_{\mathbb{X}}$. The result follows because $\mathbb{X} = \bigcup \mathsf{V}_m \approx \bigcup \mathcal{K} \approx \bigcup \operatorname{Sd} \mathcal{D}_{\mathbb{X}} = \bigcup \mathcal{D}_{\mathbb{X}}$.

To start the induction, let each $T \subseteq S$ correspond to a point v_T in $\mathbb{R}^e$. Let e be large enough and choose the points in $\mathbb{R}^e$ so that any collection of simplices of dimension up to m spanned by these points satisfy the properties of a simplicial complex. Define $\mathcal{K}_0 = \{v_T \mid T \subseteq S,\ \operatorname{card} T = m+1,\ \bigcap \mathsf{V}_{T,\mathbb{X}} \neq \emptyset\}$. At this stage the homeomorphism $\varphi_0 : \mathsf{V}_0 \to \mathcal{K}_0$ defined by $\varphi_0(\bigcap \mathsf{V}_{T,\mathbb{X}}) = v_T$ is just a bijection between two finite point sets.

Suppose $0 \leq j \leq m-1$ and $\mathcal{K}_j$ and $\varphi_j : \bigcup \mathsf{V}_j \to \bigcup \mathcal{K}_j$ are constructed. Let $i = j+1$ and initialize $\mathcal{K}_i = \mathcal{K}_j$. Let $T \subseteq S$ with $\operatorname{card} T = m+1-i$ so that $F = \bigcap \mathsf{V}_{T,\mathbb{X}} \neq \emptyset$ and v_T is not yet in $\mathcal{K}_i$. Define $G = \bigcap \mathsf{V}_{T,\operatorname{bd}\mathbb{X}}$. We add simplices to $\mathcal{K}_i$ that will allow us to extend the homeomorphism so it includes $F \in \mathsf{V}_i$. Specifically, consider all sets $U \subseteq S$, with $T \subset U$ and $\bigcap \mathsf{V}_{U,\mathbb{X}} \neq \emptyset$. Each such set U corresponds to a vertex $v_U \in \mathcal{K}_j$, and each nested sequence of such sets U corresponds to a simplex $\sigma \in \mathcal{K}_j$. These simplices σ are contained in $\varphi_j(\operatorname{bd} F - \operatorname{int} G)$. Add all simplices $\operatorname{conv}(\sigma \cup \{v_T\})$ to $\mathcal{K}_i$.

To extend the homeomorphism we distinguish two cases. Assume first that $G = \emptyset$, and therefore $F \subseteq \operatorname{int} \mathbb{X}$. Since φ_j is a homeomorphism and F is a closed i-ball, $\operatorname{bd} F$ and $\eta_j(\operatorname{bd} F)$ are both $(i-1)$-spheres. It follows that $\bigcup \operatorname{St} v_T$ is a closed i-ball, and a homeomorphism $\varphi_T : F \to \bigcup \operatorname{St} v_T$ that agrees with φ_j on $\operatorname{bd} F$ can be constructed, see figure 3. Now assume $G \neq \emptyset$. By lemma 2, $\operatorname{bd} F - \operatorname{int} G$

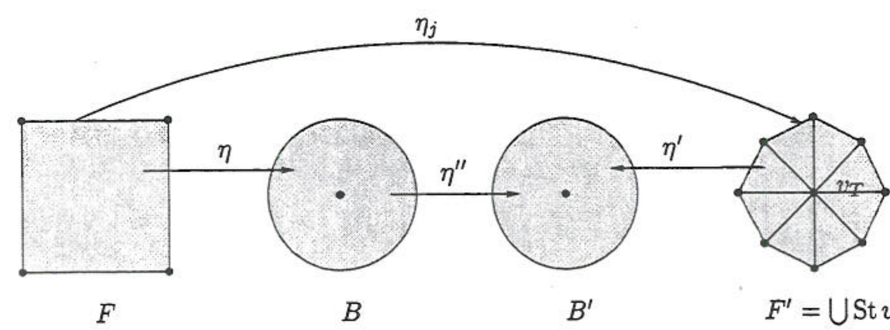

Fig. 3. φ_T is constructed from homeomorphism $\eta : F \to B$, $\eta' : F' \to B'$, and the retriction of φ_j to $\operatorname{bd} F$, $\eta_j : \operatorname{bd} F \to \operatorname{bd} F'$. B and B' are geometric closed balls of the same dimension as F and F'. By restricting η^{-1} to $\operatorname{bd} B$ we get a homeomorphism $\eta' \circ \eta_j \circ \eta^{-1} : \operatorname{bd} B \to \operatorname{bd} B'$, which can be estended to a homeomorphism $\eta'' : B \to B'$, e.g. by mapping radii of B to radii of B'. Then $\varphi_T = (\eta')^{-1} \circ \eta'' \circ \eta : F \to F'$ is a homeomorphism that agrees with φ_j on $\operatorname{bd} F$.

and therefore $\varphi_j(\operatorname{bd} F - \operatorname{int} G)$ are closed $(i-1)$-balls. Furthermore, $\operatorname{bd} G$ is an $(i-2)$-sphere.

Let $\mathcal{L} \subseteq \mathcal{K}_j$ consist of all simplices $\sigma \subseteq \varphi_j(\operatorname{bd} G)$. $\bigcup \mathcal{L}$ is an $(i-2)$-sphere and $\bigcup \mathcal{L}'$, with $\mathcal{L}' = \{\text{conv}(\sigma \cup \{v_T\}) \mid \sigma \in \mathcal{L}\}$, is a closed $(i-1)$-ball. A homeomorphism $\varphi'_T : G \to \bigcup \mathcal{L}'$ that agrees with φ_j on $\text{bd}\, G$ can be established by the method illustrated in figure 3. Now we are in the same situation as in the first case, and a homeomorphism $\varphi_T : F \to \bigcup \text{St}\, v_T$ that agrees with φ'_T on G and with φ_j on $\text{bd}\, F - \text{int}\, G$ can be constructed. After adding all $F = \bigcap \mathsf{V}_{T,\mathbb{X}}$ with $\text{card}\, T = m + 1 - i$ in this fashion, we get φ_i by combining all φ_T using the pasting lemma.

Observe that $\mathcal{K}$ and $\text{Sd}\, \mathcal{D}_{\mathbb{X}}$ contain a vertex for each $T \subseteq S$ with $\bigcap \mathsf{V}_{T,\mathbb{X}} \neq \emptyset$, so $\text{Vert}\, \text{Sd}\, \mathcal{D}_{\mathbb{X}} = \{\mathsf{b}_T \mid v_T \in \text{Vert}\, \mathcal{K}\}$. It follows that $f : \text{Vert}\, \mathcal{K} \to \text{Vert}\, \text{Sd}\, \mathcal{D}_{\mathbb{X}}$ defined by $f(v_T) = \mathsf{b}_T$ is a bijection. By construction of $\mathcal{K}$, if a collection of vertices belong to a common simplex in $\mathcal{K}$, then their images belong to a common simplex in $\text{Sd}\, \mathcal{D}_{\mathbb{X}}$. It follows that the simplicial map $g : \bigcup \mathcal{K} \to \bigcup \text{Sd}\, \mathcal{D}_{\mathbb{X}}$ implied by f is a homeomorphism. Therefore, $\bigcup \mathcal{K} \approx \bigcup \text{Sd}\, \mathcal{D}_{\mathbb{X}} = \bigcup \mathcal{D}_{\mathbb{X}}$, and the assertion of the theorem follows. □

REMARK. As illustrated in figure 2, the Delaunay complex restricted by $\mathbb{X}$ is related to the complex obtained from the chains of the regular complex defined by the Voronoi cells. Besides being smaller, an advantage of the Delaunay complex is that it naturally imbeds in the same space that contains $\mathbb{X}$ and S.

5. Non-manifold Spaces

In this section, we generalize the result for compact manifolds to non-manifold spaces. A simple example of such a space is the 'cross' defined as the union of two crossing line segments. The requirements will automatically exclude spaces that cannot be expressed as the underlying space of a finite regular complex. In order to generalize theorem 2, we need extensions of the generic intersection and the closed ball properties. Let $\mathbb{X} \subseteq \mathbb{R}^d$ be a topological space and let $S \subseteq \mathbb{R}^d$ be a non-degenerate finite point set. S has the *extended closed ball property* for $\mathbb{X}$ if there is a finite regular complex $\mathcal{R}$, with $\mathbb{X} = \bigcup \mathcal{R}$, that satisfies the following properties for every $T \subseteq S$ with $\bigcap \mathsf{V}_{T,\mathbb{X}} \neq \emptyset$:

(E1) there is a regular complex $\mathcal{R}_T \subseteq \mathcal{R}$ so that $\bigcap \mathsf{V}_{T,\mathbb{X}} = \bigcup \mathcal{R}_T$,

(E2) the set $\mathcal{R}^\circ_T = \{\gamma \in \mathcal{R} \mid \text{int}\, \gamma \subseteq \text{int} \bigcap \mathsf{V}_T\}$ contains a unique cell, τ_T, so that $\tau_T \subseteq \gamma$ for every $\gamma \in \mathcal{R}^\circ_T$,

(E3) if τ_T is a j-cell then $\tau_T \cap \text{bd} \bigcap \mathsf{V}_T$ is a $(j-1)$-sphere, and

(E4) for each integer k and each k-cell $\gamma \in \mathcal{R}^\circ_T - \{\tau_T\}$, $\gamma \cap \text{bd} \bigcap \mathsf{V}_T$ is a closed $(k-1)$-ball.

We call τ_T the *hub* of $\mathcal{R}^\circ_T$. Furthermore, S has the *extended generic intersection property* for $\mathbb{X}$ if for every $T \subseteq S$ and every $\gamma \in \mathcal{R}_T - \mathcal{R}^\circ_T$ there is a $\delta \in \mathcal{R}^\circ_T$ so that $\gamma \subseteq \delta$.

It is not difficult to see that if $\mathbb{X}$ is a compact manifold and S has the generic intersection and the closed ball properties for $\mathbb{X}$, then the extended generic intersection and the extended closed ball properties follow. Indeed, the regular complex, $\mathcal{R}$, required by condition (E1) consists of all non-empty sets $\bigcap \mathsf{V}_{T,\mathbb{X}}$ and $\bigcap \mathsf{V}_{T,\mathrm{bd}\,\mathbb{X}}$, $T \subseteq S$. Fix a subset $T \subseteq S$ and define $F = \bigcap \mathsf{V}_{T,\mathbb{X}}$ and $G = \bigcap \mathsf{V}_{T,\mathrm{bd}\,\mathbb{X}}$. If non-empty, F and G are closed balls, and if $G \neq \emptyset$ then $F \neq \emptyset$ and the dimension of F exceeds the dimension of G by one. There are three possible cases. If $F = G = \emptyset$ then $\bigcap \mathsf{V}_{T,\mathbb{X}} = F = \emptyset$ and conditions (E2) through (E4) are void. If $F \neq \emptyset$ and $G = \emptyset$ then $\mathcal{R}_T^\circ = \{F\}$, and $\tau_T = F$ satisfies conditions (E2) and (E3); condition (E4) is void. If $F \neq \emptyset$ and $G \neq \emptyset$ then $\mathcal{R}_T^\circ = \{F, G\}$, $\tau_T = G$ satisfies conditions (E2) and (E3), and $\gamma = F$ satisfies condition (E4). In any case, the establishment of the homeomorphism in the proof of theorem 2 can be viewed as introducing a vertex v_T for τ_T and connecting it to the simplices inductively constructed for the cells in $\mathcal{R}_T - \mathcal{R}_T^\circ$. This idea also works in the general case considered in this section.

Let $\mathbb{X} \subseteq \mathbb{R}^d$ be a topological space, and let $S \subseteq \mathbb{R}^d$ be a non-degenerate finite point set that has the extended generic intersection and the extended closed ball properties for $\mathbb{X}$. To construct a homeomorphism between $\mathbb{X} = \bigcup \mathcal{R}$ and $\bigcup \mathcal{D}_{\mathbb{X}}$, we consider one subset $T \subseteq S$ at a time, in order of non-increasing cardinality. Inductively, we assume the cells in $\mathcal{R}_T - \mathcal{R}_T^\circ$ are already mapped homeomorphically to appropriate simplices. We extend the homeomorphism to the cells of $\mathcal{R}_T^\circ$, again inductively in order of non-decreasing dimension. We introduce a vertex, v_T, for the hub, τ_T. If τ_T is a k-ball, its boundary is a $(k-1)$-sphere, and by (E3) and the induction hypothesis, the cells in this $(k-1)$-sphere are already part of the homeomorphism. After connecting v_T to the simplices that correspond to the cells in $\mathrm{bd}\,\tau_T$, we can extend the homeomorphism to τ_T. Every other cell in $\mathcal{R}_T^\circ$ is treated the same way, only that the reason the homeomorphism can be extended is now a combination of the two induction hypotheses. This implies the generalization of theorem 2 to topological spaces other than manifolds.

Theorem 3 *Let $\mathbb{X} \subseteq \mathbb{R}^d$ be a topological space, and let S be a non-degenerate finite point set that has the extended generic intersection property for $\mathbb{X}$. If S has the extended closed ball property for $\mathbb{X}$ then $\bigcup \mathcal{D}_{\mathbb{X}} \approx \mathbb{X}$.*

6. Remarks and Further Work

This paper is targeted at problems requiring the discretization of possibly complicated geometric objects in finite-dimensional Euclidean spaces. Such problems are abundant in the computational science literature, see e.g. Kaufmann and Smarr,[11] where the discretization of continuous domains is common practice. The dimensions of the domain and the imbedding space can be the same, as e.g. common in fluid dynamics problems, or they can be widely different, as in the study of many dynamical systems. The restricted Delaunay complex introduced in this paper is a general method that produces simplicial discretizations in all cases.

The introduction of a general concept typically gives rise to many specific questions and directions for further research. We see three directions of progressively more basic work necessary to bring restricted Delaunay complexes closer to the targeted application areas. The first is the design of efficient algorithms that constructs the Delaunay complex of a finite point set, S, restricted by a domain or space, $\mathbb{X}$. Special cases under different assumptions on how $\mathbb{X}$ is specified are considered in Ref. [3,7,14]. The second direction is the design of methods that choose finitely many points resulting in good quality discretizations of $\mathbb{X}$. Such methods have a long history in the somewhat different contexts of finite element analysis[20] and free-form modeling.[22] We remark that the method by Welch and Witkin[22] uses the algorithm due to Chew[3] to connect a chosen point set to form a surface; it can thus be seen as an application of the idea of restricted Delaunay complexes. The third direction is the study of maps from a problem specific domain to a space, possibly imbedded in higher dimensions, that depends on functionals studied over the domain or on approximate solutions thereof.

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