Auditory Morse Analysis of Triangulated Manifolds

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Abstract. Visualization of high-dimensional or large geometric data sets is inherently difficult, so we experiment with the use of audio to display the shape and connectivity of these data sets. Sonification is used as both an addition to and a substitution for the visual display. We describe a new algorithm called *wave traversal* that provides a necessary intermediate step to sonification of the data; it produces an ordered sequence of subsets, called *waves*, that allows us to map the data to time. In this paper we focus in detail on the mathematics of wave traversal, in particular, how wave traversal can be used as a discrete Morse function.

1 Audio as an Experimental and Analytic Tool

Visualization has become an important tool for mathematicians, allowing them to see complicated spaces and to further their insight into the nature of these spaces. No amount of imagination can equal a visual ride through hyperbolic space, for example, as evidenced by the video *Not Knot* [6] or the CAVE™ application *postEuclidean Walkabout* [5]. Visualization is limited by our ability to see in only three dimensions, however, and by our ability to see for only short distances, often due to occlusion. The latter problem becomes particularly acute when we use a two dimensional display to visualize a three dimensional world.

These observations have led us to investigate *sonification*. Sonification is "the use of data to control a sound generator for the purpose of monitoring and analysis of the data..." [8, p. 187]. Kramer points out in [7] that the ability of sound to display multiple variables simultaneously can be used to increase the display dimensionality of a visual system, or it can be used by itself to display a high-dimensional system. We are investigating sonification of simplicial complexes of dimensions three and higher, with and without a visual display.

One problem associated with using geometric data to control an audio signal is the static nature of the data. Sound is perceived through time: it has an intrinsic time dimension. We describe in the next section an algorithm for making the data dynamic. This algorithm, called *wave traversal*, outputs an ordered sequence of subsets of the data. We then show how to make wave traversal on triangulated manifolds into a discrete Morse function. Thus wave traversal is both an intermediate step to sonification and an analytic tool. In this paper we focus on the mathematics of wave traversal, but we describe now briefly its application to sonification.

We apply maps, called *transfer functions*, from the output of the wave traversal into the audio domain. For example, the relative size of each subset in the sequence might control the carrier frequency (in Hertz) of a tone, and the number of components of each subset might be mapped to a parameter that modulates the frequency. The order of the sequence of subsets gives us a natural map to the time dimension. Critical points are mapped to specific signals so that they can be easily distinguished from sound that is controlled by other data.

Wave traversal does not solve the specific problem of how to map geometric data to parameters of a sound synthesis algorithm, however. So, we still need to supply arbitrary but meaningful maps from data to sound. These maps have been developed with collaborators from the National Center for Supercomputing Applications and refined with experimentation. Some details of the sonification can be found in [1] and [2]. User training is an important part of this process; the audio signals must first be interpreted for a new user before they become meaningful.

2 Wave Traversal

For definitions of terms from piecewise linear topology in this section, see [10], for example. Let K be a finite simplicial complex and assume K is connected. Let the distance between two vertices u and v of K be defined as the number of edges in the shortest path from u to v. We will denote this distance by $d(u, v)$ and refer to u as the start vertex. For a simplex $\sigma \in K$ with dimension greater than 0, define the distance $d(u, \sigma)$ to be equal to the minimum $d(u, v)$ over all vertices v of σ.

Definition 1. Let $W_{K,u}(i), i \geq 0$, be the set of $\sigma \in K$ such that

1. $d(u, v) = i$ for all vertices $v \in \sigma$, and
2. σ is face of some simplex $\tau \in K$ with $d(u, \tau) = i - 1$.

For example, $W_{K,u}(0) = \{u\}$. Condition 1 implies that waves are progressively further from u as the index i increases. Condition 2 guarantees that each wave has dimension strictly less than the dimension of K. We will assume that the complex K and the start vertex u are fixed, and simplify our notation to $W(i) = W_{K,u}(i)$. Let D be the maximum $d(u, v)$ for all $v \in K$ and define $W = \{W(i) \mid 0 \leq i \leq D\}$. We call the algorithm that produces W *wave traversal* because of the analogy to a wave moving through a medium, see Fig. 1. We will refer to $W(i)$ as a *wave* and W as the *wave subcomplex*.

Let $S = S_K$ be the set of simplices of K that do not belong to W. Def. 1 implies that all vertices of K are in W, so S contains only simplices of dimension 1 or higher. Let $S(i)$ be the set of simplices in S such that $d(u, \sigma) = i$.

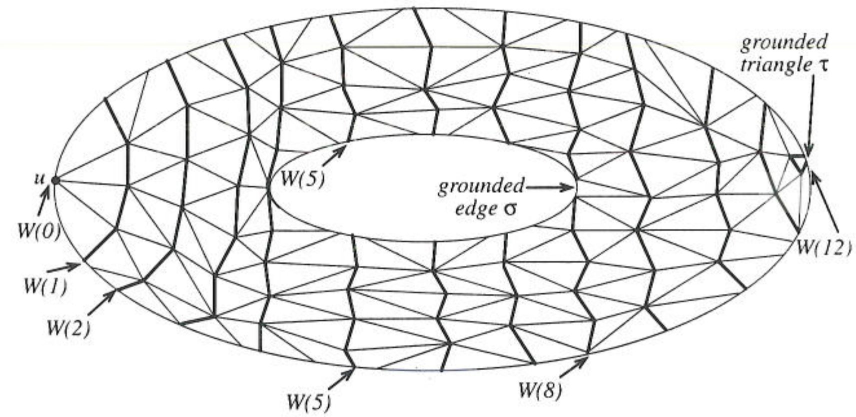

Fig. 1. Waves on a 2-dimensional torus. $W(8)$ includes both vertices of edge σ but not σ itself. Therefore, σ is a grounded edge in $S(8)$. $W(12)$ includes all the edges of triangle τ but not τ itself, so τ is a grounded triangle in $S(12)$.

We would like to extend $d(u, v)$ linearly to a continuous function $d(u, x)$ over all points of K, so that $d(u, x) = i$ for all points of $\sigma \in W(i)$, and $d(u, x)$ varies from i to $i+1$ for all points of $\sigma \in S(i)$. For most $\sigma \in S(i)$ this presents no problem, since some $v \in \sigma$ belong to $W(i)$ and some belong to $W(i+1)$. However, it is possible to have a simplex in $S(i)$ which has all vertices in $W(i)$. We call such a simplex a *grounded simplex*. This can happen, for example, when $W(D)$ is the boundary of a simplex of maximal dimension, or when $W(i)$ consists of two components which divide all the vertices of a simplex in $S(i)$ between them, see Fig. 1. Note that in these cases, grounded simplices occur at critical places where the connectivity of wave $W(i+1)$ is different from wave $W(i)$. We will investigate this relationship further in Sec. 4.

Let $\sigma_i \prec \sigma_j$ denote σ_i is a proper face of σ_j. We consider the *barycentric subdivision* sd$K$ of K, whose vertices are the barycenters b_σ of the simplices $\sigma \in K$. sd$K$ is the collection of all simplices of the form $b_{\sigma_0} b_{\sigma_1} \ldots b_{\sigma_k}$, where $\sigma_0 \prec \sigma_1 \prec \ldots \prec \sigma_k$. Let $W_{\text{sd}K}$ be the wave subcomplex of sd$K$, and let $S_{\text{sd}K}$ be defined accordingly. The start vertex u is defined to be the same for both W_K and $W_{\text{sd}K}$. Somewhat surprisingly, we can eliminate grounded simplices from the wave traversal using barycentric subdivision, and we state this as a theorem and prove this in the rest of this section.

Theorem 1. All vertices of a simplex $\sigma \in$ sd$K$ have distance i if and only if $\sigma \in W_{\text{sd}K}(i)$. That is, sd$K$ contains no grounded simplices.

We first prove a lemma which says that the distances in K are doubled in sd$K$.

Lemma 2. *For vertices u and v in K, $d(b_u, b_v) = 2d(u, v)$.*

Proof. We will show that there is a shortest path from b_u to b_v in sd$K$ that is restricted to sd$(K^{(1)})$, where $K^{(1)}$ is the 1-skeleton consisting of all edges and vertices of K. The lemma then follows. Let fd(b_σ) (for *former dimension* of b_σ) be equal to dim(σ). We want to show that there is a shortest path from b_u to b_v in sd$K$ whose vertices b_σ have only fd$(b_\sigma) = 0$ or 1. Note that fd$(b_\sigma) \neq$ fd(b_τ) if b_σ is adjacent to b_τ. Also, no 3 contiguous vertices in the shortest path, b_ρ, b_σ, b_τ, can have fd$(b_\rho) <$ fd$(b_\sigma) <$ fd(b_τ), since this would imply that $\rho \prec \sigma \prec \tau$, and so there would be an edge from b_ρ to b_τ. Similarly we cannot have fd$(b_\rho) >$ fd$(b_\sigma) >$ fd(b_τ). Thus, any shortest path from b_u to b_v must have a sequence of former dimensions of the form

$$0 < \mathrm{fd}(b_1) > \mathrm{fd}(b_2) < \mathrm{fd}(b_3) > \cdots < \mathrm{fd}(b_n) > 0 \ .$$

fd(b_2), fd$(b_4), \ldots,$ fd(b_{n-1}) are all locally minimum in this sequence, and so we can replace the vertices $b_2, b_4, \ldots, b_{n-1}$ with original vertices of K to get a path of the same length with

$$0 < \mathrm{fd}(b_1) > 0 < \mathrm{fd}(b_3) > \cdots < \mathrm{fd}(b_n) > 0$$

(see Fig. 2). Now vertices $b_1, b_3, \ldots, b_n$ are all between two original vertices, and so can be replaced by the barycenters of the edges connecting those vertices in K. □

Lemma 2 suggests that every other wave in sd$K$ is also a wave in K, or rather the barycentric subdivision of one. We present a technical lemma before proving that this is indeed the case. Call a simplex $\Sigma \in K$ that has $d(u, v) = 2i$ in sd$K$ for all vertices $v \in \Sigma$ a *dip*, a *flat*, or a *bump* if $d(u, b_\Sigma) = 2i - 1$, $2i$, or $2i + 1$, respectively.

Lemma 3. *1. K contains no dips.*
2. If $\Sigma \in K$ is a flat then $d(u, w)$ is the same for every vertex $w \in$ sdΣ.

Proof. 1. Suppose K contains a dip Σ. Then $d(u, b_\Sigma) = 2i - 1$ and $d(u, x) = 2i - 2$ for the predecessor of b_Σ along some shortest path from u to b_Σ. If x is a vertex of sdΣ then it is connected by an edge to at least one vertex $v \in \Sigma$, and if $x \notin$ sdΣ then it is the barycenter of a coface [1] of Σ and therefore connected by an edge to every vertex $v \in \Sigma$. In either case we have a contradiction because $d(u, x) = 2i - 2$ and $d(u, v) = 2i$.

2. We have $d(u, b_\Sigma) = 2i$ if Σ is a flat in K. By claim 1, no vertex $w \in$ sdΣ has $d(u, w) = 2i - 1$, because then $w = b_T$ for a face T of Σ and T would be a dip. So, $d(u, w) \geq 2i$. Let x again be the predecessor of b_Σ along a shortest path from u. $d(u, x) = 2i - 1$ and $x \notin$ sdΣ. Therefore x is the barycenter of a coface of Σ and thus connected by an edge to every vertex $w \in$ sdΣ. So $d(u, w) \leq 2i$. Therefore $d(u, w) = 2i$ as claimed. □

[1] A *coface* of Σ is a simplex which has Σ as a face.

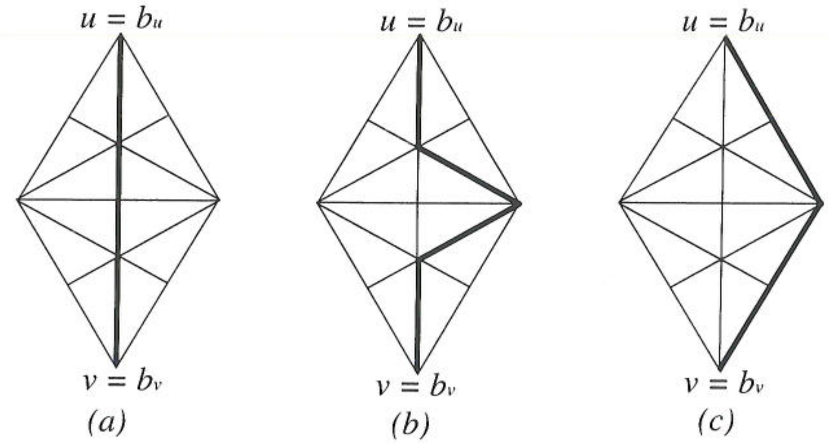

Fig. 2. Shortest paths from b_u to b_v. In (a), the path goes through the barycenters of the two triangles and their common edge. The fd sequence is $0 < 2 > 1 < 2 > 0$. Since the barycenter of the edge has former dimension which is a local minimum, we can replace it with an original vertex to get (b), with fd sequence of $0 < 2 > 0 < 2 > 0$. Finally, we can replace the barycenters of the two triangles, now between original vertices, with barycenters of edges to get (c), a shortest path contained entirely in sd$(K^{(1)})$.

By Lemma 3, a simplex in K whose vertices all have the same distance from u is either a bump, or a flat whose faces are also flats. Lemma 3 is used in the proof that every other wave in sd$K$ is the barycentric subdivision of a wave in K.

Lemma 4. sd$W_K(i) = W_{\mathrm{sd}K}(2i)$.

Proof. First we show $W_{\mathrm{sd}K}(2i) \subseteq$ sd$W_K(i)$. Let σ be a k-simplex in $W_{\mathrm{sd}K}(2i)$ and let $\Sigma \in K$ be the simplex of lowest dimension that contains σ. By definition, $d(u, v) = 2i$ for all $v \in \sigma$; in particular, $d(u, b_\Sigma) = 2i$. All original vertices $w \in \Sigma$ are adjacent to b_Σ, and so must have distance $2i - 1$, $2i$, or $2i + 1$. Lemma 2 implies that $d(u, w)$ must be $2i$ in sd$K$, and so $d(u, w) = i$ in K. In other words, Σ is a flat.

Now, σ is face of a simplex $\tau \in$ sd$K$ with $d(u, \tau) = 2i - 1$. We may assume that τ has dimension $k + 1$. Let T be the simplex of lowest dimension in K that contains τ. Since Σ is a flat it cannot contain τ and therefore $\Sigma \prec T$. Also note that the vertex of τ at distance $2i - 1$ is the barycenter b_T. The predecessor x of b_T along a shortest path from u to b_T has distance $2i - 2$. Assume that no vertex of T has distance $2i - 2$. Then x is not one of the vertices of T and so it must be the barycenter of a coface of T or of a face of T. If x is the barycenter of a coface of T, then x is adjacent to a vertex v of Σ, implying that $d(u, v) < 2i$, a contradiction. If x is the barycenter of a face of T, then at least one vertex w of this face has distance $2i - 2$, or else that face is a dip, which would contradict Lemma 3(1). But $d(u, w) = 2i - 2$ is also a contradiction. Thus x must be a vertex of T and $d(u, T) = i - 1$ in K.

Now we will show that sd $W_K(i) \subseteq W_{\mathrm{sd}K}(2i)$. Let Σ be a k-simplex in $W_K(i)$. Σ is face of a $(k+1)$-simplex T with $d(u, T) = i - 1$. The vertices of Σ have distance $2i$ in sd$K$ and the other vertex of T has distance $2i - 2$. Since b_T is adjacent to all of these, $d(u, b_T) = 2i - 1$. All $(k+1)$-simplices $\tau \in$ sd T with a k-face in sd Σ thus have distance $2i - 1$. By Lemma 3(1), the barycenter of Σ cannot have distance $2i - 1$ and it has distance at most $2i$ because it is adjacent to b_T. So, by Lemma 3(2), all vertices of sdΣ have distance $2i$ from u. Hence all simplices σ in sd$K$ contained in Σ satisfy both conditions for belonging to wave $2i$ of sd$K$. □

It follows from Lemma 4 that the barycenters of all simplices in S_K belong to odd-numbered waves of sd$K$. We can show a stronger result.

Lemma 5. *The barycenters of all simplices in $S_K(i)$ belong to $W_{\mathrm{sd}K}(2i+1)$.*

Proof. If $\Sigma \in S_K(i)$ has vertices at distance i and $i+1$ then $d(u, b_\Sigma) = 2i+1$ in sd$K$ and b_Σ belongs to $W_{\mathrm{sd}K}(2i+1)$ as claimed. The only other case is that Σ is grounded, that is, all its vertices have distance $2i$ in sd$K$. By Lemma 3(1) we have $d(u, b_\Sigma) \neq 2i - 1$, and by Lemma 4 we have $d(u, b_\Sigma) \neq 2i$. The only remaining possibility is $d(u, b_\Sigma) = 2i + 1$. □

Now we have the tools to prove Theorem 1, which says that if we first take the barycentric subdivision of a complex, we can eliminate the second condition of Def. 1; in sd$K$ equidistance of vertices is the sole determining condition for inclusion of a simplex into a wave.

Proof (of Theorem 1). We show that the assumption that $\sigma \in$ sd$K$ is grounded leads to a contradiction. Let Σ be the simplex of lowest dimension in K that contains σ; b_Σ is a vertex of σ. Σ cannot belong to a wave of K, otherwise Lemma 4 implies that σ belongs to a wave of sd$K$ and can therefore not be grounded.

We may therefore assume that $\Sigma \in S_K(j)$. By Lemma 5, $d(u, b_\Sigma) = 2j+1$. For σ to be grounded it must have all vertices at distance $2j + 1$. Besides σ, sdΣ also contains simplices τ that contain σ and one original vertex v of Σ. If $d(u, v) = 2j$ for any such τ then σ is in a wave and cannot be grounded, a contradiction. If $d(u, v) = 2j + 2$ for all choices of τ then we have a face of Σ that is a dip; its vertices are the vertices v of the simplices τ, and its barycenter is a vertex of σ. This contradicts Lemma 3(1).

Since $S_{\mathrm{sd}K}$ contains no grounded simplices, we may conclude that if all vertices of $\sigma \in$ sd$K$ have distance i, then $\sigma \in W_{\mathrm{sd}K}(i)$. In other words, we do not need the second condition of Def. 1. □

Figures 3 and 4 illustrate the cases where $\Sigma \in K$ is a triangle not contained in $W(i)$ for any i. In Fig. 3 Σ is a grounded simplex, and in Fig. 4 it is not.

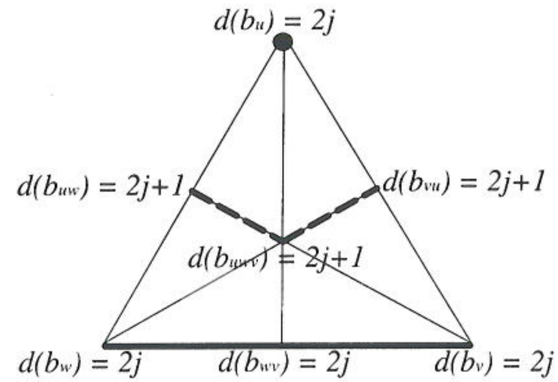

Fig. 3. Before subdivision, vertex u and the segment wv were in $W(j)$, while triangle $\Sigma = uwv$ was in $S(j)$, a grounded simplex. After barycentric subdivision, all simplices with equidistant vertices belong to W.

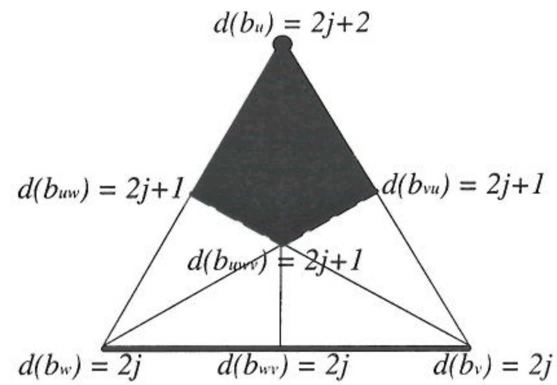

Fig. 4. Before subdivision, vertex u was in $W(j+1)$ and the segment wv was in $W(j)$, while triangle $\Sigma = uwv$ was in $S(j)$, *not* a grounded simplex. After barycentric subdivision, the white triangles belong to $S(2j)$ and the black ones belong to $S(2j+1)$.

3 Wave Traversal as a Morse Function

Morse theory relates the critical points of a smooth function on a smooth manifold to the connectivity of the manifold; details of Morse theory can be found in [9]. A smooth function that is often used for Morse analysis is a height function $h : M \to \mathbb{R}$ that maps a point of M to its distance from a hyperplane, see Fig. 5. In this section, we will construct a piecewise linear function f analogous to the smooth height function h, and show how to isolate the critical points of f so that we obtain a discrete Morse function.

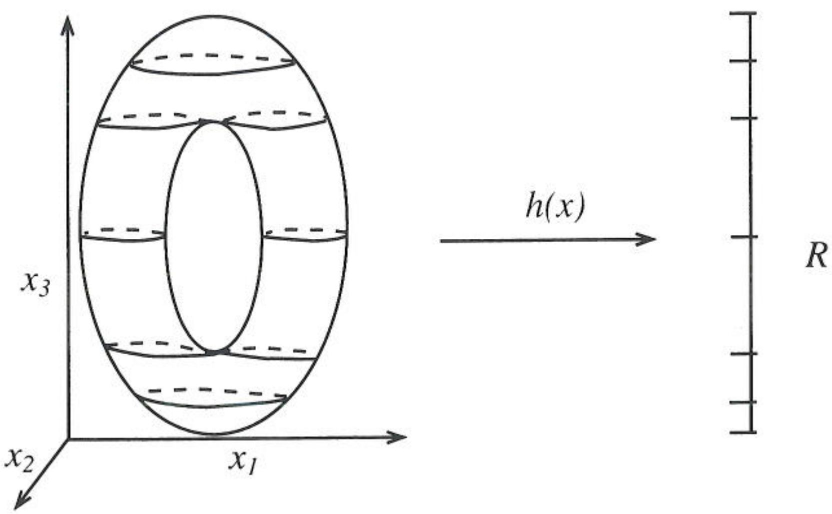

Fig. 5. $h : M \to \mathbb{R}$ takes points of the 2-torus to real numbers, $(x_1, x_2, x_3) \to x_3$.

In classical Morse theory, critical points of h are found by examining the gradient of h, and the index of each critical point is found by examining the matrix of second derivatives of h evaluated at the critical point p. If this matrix has full rank, the critical point p is non-degenerate. For our purposes, though, we are interested in a geometric interpretation of critical points and their indices, and we will present these as Banchoff does in [3].

Assume M is a smooth 2-manifold in $\mathbb{R}^3$. If $h : M \to \mathbb{R}$ is a Morse function, then critical points of h are isolated and have only three types: minima, saddle points, and maxima, of index 0, 1, and 2, respectively. If p is a critical point for h then the tangent plane to M at p is horizontal. Consider a small circle about p on M. If p is a minimum or maximum, the plane through p does not intersect the circle. If p is a saddle point, it intersects the circle in 4 points. Another way to describe this is that the tangent plane at a saddle point p divides a small disk neighborhood of p on M into "four separate pieces." The horizontal plane through a *regular* or non-critical point q is not the tangent plane, and therefore meets a small circle about q in two points, and divides a small disk neighborhood of q on M into two pieces.

We will use these observations to find critical points of our discrete function. In the following we assume that K is a triangulated oriented closed surface; i.e., K is homeomorphic to an oriented compact 2-manifold. We also assume that K is embedded in $\mathbb{R}^3$, although what follows is applicable to abstract manifolds as well. Finally, we assume now that K is the first barycentric subdivision of another simplicial complex, so that we can apply the results of the previous section, in particular Theorem 1.

We extend the distance function for the vertices linearly to all points of K. Since K contains no grounded simplices, we do not lose the correspondence between distance and waves. Let $x \in \sigma$ and let $t_v = t_v(x)$ be its *barycentric coordinates*, where $t_v = 0$ if $v \notin \sigma$,

$$1 = \sum_v t_v \ , \text{ and } x = \sum_v t_v \cdot v \ .$$

Definition 6. The distance from u to x is

$$d(u, x) = \sum_v t_v \cdot d(u, v) \ .$$

Observe that $d(u, x)$ is defined over the entire set of points of K and is continuous. If we embed K in $\mathbb{R}^4$ so that

$$x = (x_1, x_2, x_3) \to (x_1, x_2, x_3, d(u, x)) \ ,$$

then $f(x) = d(u, x)$ becomes a height function.

Lemma 7. *All critical points of f are in the wave subcomplex W.*

Proof. We will show that all points $x \in |S|$, where $|S| = \bigcup_{\sigma \in S} \operatorname{int} \sigma$, are regular. Consider a point $x \in \operatorname{int} \sigma$, with $\sigma \in S$. Since grounded simplices have been eliminated, σ has two vertices at different heights and is therefore not contained in the horizontal hyperplane passing through x. It follows that this hyperplane cuts a suitable neighborhood of x into precisely two pieces, compare [3, p. 478]. □

Observe that each wave $W(i)$ is in a horizontal hyperplane in $\mathbb{R}^4$. A vertex $v \in W(i)$ is a regular point of f if $W(i)$ cuts the star of v, $\mathrm{St}(v)$, into exactly two pieces, one a subset of $S(i-1)$ and the other a subset of $S(i)$ (Fig. 6). Similarly, all points x of an edge $e \in W(i)$ are regular if $W(i)$ divides $\mathrm{St}(e)$ into two pieces, where $\mathrm{St}(e)$ contains e and the two triangles that share e.

f has only one minimum and it is non-degenerate; it is the start vertex u. All simplices in $\mathrm{St}(u)$, except u, belong to $S(0)$. Similarly, for an isolated local maximum $v \in W(i)$, $\mathrm{St}(v)$ consists only of v and simplices in $S(i-1)$.

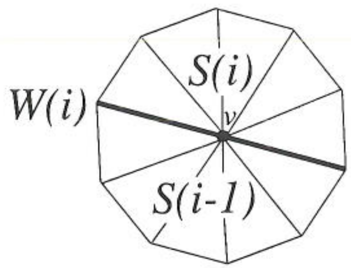

Fig. 6. The star of a regular point $v \in W(i)$. $W(i)$ divides the star into two pieces, one a subset of $S(i-1)$ and the other a subset of $S(i)$.

If $v \in W(i)$ is an isolated saddle point, then $W(i)$ divides $\mathrm{St}(v)$ into 4 pieces, alternating between $S(i-1)$ and $S(i)$ (Fig. 7(*a*)). It is also possible that several saddle points come together at a single location. The star of an m-fold saddle point is cut by $W(i)$ into $2m+2$ pieces. A 2-fold saddle point is also know as a "monkey" saddle.

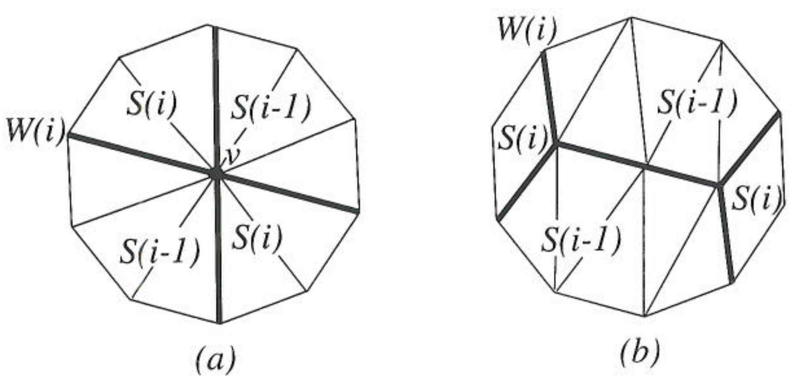

Fig. 7. (*a*) illustrates the star of an isolated saddle point v. (*b*) illustrates a component of a critical subcomplex.

Critical points of f are not isolated in general, so we need a way to identify and resolve these degeneracies. A degeneracy occurs when one or more adjacent edges $e \in W(i)$ all have $\mathrm{St}(e) \subseteq W(i) \cup S(i-1)$, see for example Fig. 7(*b*). More formally, we define a *(degenerate) critical subcomplex* C of $W(i)$ as the closure of all edges e such that $\mathrm{St}(e) \subseteq W(i) \cup S(i-1)$. Note that it is possible for some vertices in C to have stars which also contain simplices in $S(i)$.

Since C is itself a simplicial complex, wave traversal is defined for its components and we use it to isolate critical points. We choose an arbitrary vertex w in a component C_p of C and compute $d(w,x)$ on C_p. [2] Then we replace f on C_p with

$$f(x) = d(u,w) - \varepsilon \cdot d(w,x) \ , \tag{1}$$

for small $\varepsilon > 0$. After redefining f on critical subcomplexes, we need to recalculate f on the rest of K.

Theorem 2. Let x and y be two vertices of C_p connected by an edge e. Then $f(x) \neq f(y)$, i.e., (1) isolates critical points of f in C.

Proof. Suppose $d(w,x) = d(w,y) = j$. Then edge e is in wave j of the wave traversal on C_p, since otherwise it would be grounded. But C_p is a 1-dimensional complex, so wave j must be 0-dimensional and cannot therefore contain e. Therefore $d(w,x) \neq d(w,y)$ and as a consequence $f(x) \neq f(y)$. □

We now illustrate with some examples. Suppose C_p comprises an entire connected component of wave $W(i)$. Then $\mathrm{St}(v) \subseteq W(i) \cup S(i-1)$ for all $v \in C_p$ because K is a manifold. The new start vertex w becomes a local maximum for f. If C_p contains no closed curve, then it contains no further critical point. If C_p contains a closed curve, then (1) isolates both a saddle point and a maximum. This happens, for example, when $W(D)$ is a (homologically) non-trivial closed curve on the torus, see Fig. 8. We prove this in the following claim.

Claim 8. *Let C_p comprise an entire connected component of $W(i)$ and suppose C_p contains a closed curve P. (1) isolates both a maximum and a saddle point on C_p.*

Proof. We recalculate f on C_p with (1) and as in the general case, w is an isolated maximum. Since there are no grounded simplices in the wave traversal on C_p, there is a vertex x on P which has a minimum value for f on P. This vertex x becomes an isolated saddle point, because f increases from x in two directions on C_p, and f decreases in two directions in $S(i-1)$. □

In fact, it is clear from the proof that (1) will isolate one saddle point for *each* closed curve in C_p. To give an intuitive understanding of why this must be, we prove the following claim.

Claim 9. *Let C_p be as in Claim 8. The closed curve P in C_p must be non-trivial.*

[2] It is a fact that there are no grounded simplices in this wave traversal, but we will not prove it here.

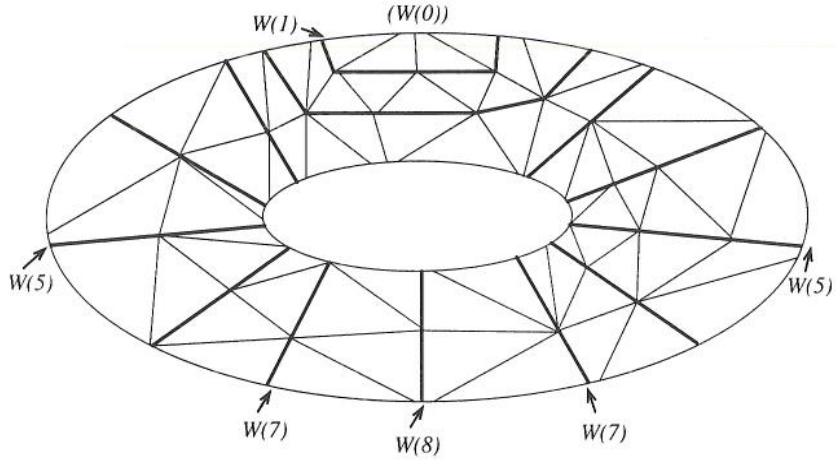

Fig. 8. $W(8)$ has a maximum and a saddle point.

Proof. Assume that P is a trivial closed curve on the surface K. P divides K into two pieces, only one of which contains the start vertex u. Any path from u to a vertex x in the other piece, and in particular the shortest path, must go through P, and so $d(u, x) = i + 1$. Therefore $\mathrm{St}(v)$ for some $v \in C_p$ must have simplices in $S(i)$, a contradiction. If the piece that does not contain u does not contain any vertices at all, then it must contain a grounded simplex, which is also impossible. □

Now suppose C_p is a subcomplex of a larger connected component of $W(i)$, see Fig. 7(b). In this figure, C_p consists of two edges and their three endpoints. The two outer endpoints include triangles from $S(i)$ in their stars. Call these vertices x and y and suppose w is different from both. After recalculating f on C_p, w again becomes an isolated maximum and both x and y become isolated saddle points. If $w = x$, then y becomes the only isolated critical point on C_p; it is a saddle point. Since a maximum and a saddle point effectively cancel each other out, the net result in both cases is a saddle point.

When only one vertex $v \in C_p$ has simplices in $S(i)$, then a similar case analysis for w shows that the critical points isolated by (1) all cancel; this is analogous to the "shoe" saddle on a continuous manifold.

4 Computation of Waves, Critical Points and Sound

Computing the waves on $\mathrm{sd}K$ is straightforward. We find $f(v) = d(u, v)$ for each vertex using *breadth-first search* [4]. We alter this algorithm slightly because we do not compute $\mathrm{sd}K$ explicitly first; we perform the breadth-first

search on an implicit subdivision of K. We add a simplex to $W(i) = W_{\mathrm{sd}K}(i)$ if all its vertices have $f(v) = i$.

We locate critical subcomplexes in odd-numbered waves by locating grounded simplices of the original complex K. It is a fact that the barycenter b_Σ of a grounded simplex Σ is either an isolated critical point of f, or it is in a degenerate critical subcomplex. For example, in Fig. 3 the barycenter of the triangle is in a degenerate critical subcomplex. If Σ is not grounded, then b_Σ is a regular point of f, see Fig. 4. In even-numbered waves, we analyze the star of each simplex to locate critical subcomplexes.

We find the connected components of critical subcomplexes and identify them using the cases illustrated in Sec. 3. For example, if a critical subcomplex C_p contains no closed curves and $\mathrm{St}(v) \subseteq W(i) \cup S(i-1)$ for all $v \in C_p$, then C_p contains a local maximum for f and no other critical points.

To compute the sound, we map properties of waves and critical points dynamically to parameters of a sound synthesis algorithm, so that we listen to the process (the wave traversal) as well as the analysis (the result of the Morse analysis). The sonification of the wave traversal itself conveys to a trained user some of the features of each wave, for example, its relative size and number of components. Critical points are mapped to specific sound signals, so that they can be distinguished from the underlying bed of sound produced by the wave traversal. These signals indicate the type and number of critical points found at a particular wave step. The user then gains knowledge of the shape and connectivity of the data by listening to the entire composition. The ideas in this paper are adapted to sonification of general 2- and 3-complexes.

The algorithms are implemented in C++ on the SGI platform, with a visual display for complexes of dimension 3 or less. A program called vss (for vanilla sound server) provides the sound synthesis algorithms and an interface for controlling these algorithms with our data (see http://www.ncsa.uiuc.edu/-VEG/audio).

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