Area and Perimeter Derivatives of a Union of Disks ⋆

Ho-Lun Cheng and Herbert Edelsbrunner

Department of Computer Science, National University of Singapore, Singapore.
Department of Computer Science, Duke University, Durham,
and Raindrop Geomagic, Research Triangle Park, North Carolina.

Abstract. We give analytic inclusion-exclusion formulas for the area and perimeter derivatives of a union of finitely many disks in the plane.

Keywords. Disks, Voronoi diagram, alpha complex, perimeter, area, derivative.

1 Introduction

A finite collection of disks covers a portion of the plane, namely their union. Its size can be expressed by the area or the perimeter, which is the length of the boundary. We are interested in how the two measurements change as the disks vary. Specifically, we consider smooth variations of the centers and the radii and study the derivatives of the measurements.

We have two applications that motivate this study. One is topology optimization, which is an area in mechanical engineering [1, 2]. Recently, we began to work towards developing a computational representation of skin curves and surfaces [7] that could be used as changing shapes within a topology optimizing design cycle. Part of this work is the computation of derivatives. The results in this paper solve a subproblem of these computations in the two-dimensional case. The other motivating problem is the simulation of molecular motion in molecule dynamics [9]. The setting is in three-dimensional space, and the goal is to simulate the natural motion of biomolecules with the computer. The standard approach uses a force field and predicts changes in tiny steps based on Newton's second law of motion. The surface area and its derivative are important for incorporating hydrophobic effects into the calculation [4].

The main results of this paper are inclusion-exclusion formulas for the area and the perimeter derivatives of a finite set of disks. As it turns out, the area derivative is simpler to compute but the perimeter derivative is more interesting. The major difference between the two is that a rotational motion of one disk about another may have a non-zero contribution to the perimeter derivative while it has no contribution to the area derivative.

Outline. Section 2 introduces our approach to computing derivatives and states the results. Section 3 proves the result on the derivative of the perimeter. Section 4 proves the result on the derivative of the area. Section 5 concludes the paper.

⋆ Research by both authors was partially supported by NSF under grant DMS-98-73945 and CCR-00-86013.

2 Approach and Results

In this section, we explain how we approach the problem of computing the derivatives of the area and the perimeter of a union of finitely many disks in the plane.

Derivatives. We need some notation and terminology from vector calculus to talk about derivatives. We refer to the booklet by Spivak [10] for an introduction to that topic. For a differentiable map $f : \mathbb{R}^m \to \mathbb{R}$, the derivative at a point $\mathbf{z} \in \mathbb{R}^m$ is a linear map $\mathrm{D}f_{\mathbf{z}} : \mathbb{R}^m \to \mathbb{R}$. The geometric interpretation is as follows. The graph of $\mathrm{D}f_{\mathbf{z}}$ is an m-dimensional linear subspace of $\mathbb{R}^{m+1}$. The translation that moves the origin to the point $(\mathbf{z}, f(\mathbf{z}))$ on the graph of f moves the subspace to the tangent hyperplane at that point. Being linear, $\mathrm{D}f_{\mathbf{z}}$ can be written as the scalar product of the variable vector $\mathbf{t} \in \mathbb{R}^m$ with a fixed vector $\mathbf{u}_{\mathbf{z}} \in \mathbb{R}^m$ known as the *gradient* of f at $\mathbf{z}$: $\mathrm{D}f_{\mathbf{z}}(\mathbf{t}) = \langle \mathbf{u}_{\mathbf{z}}, \mathbf{t} \rangle$. The *derivative* $\mathrm{D}f$ maps each $\mathbf{z} \in \mathbb{R}^m$ to $\mathrm{D}f_{\mathbf{z}}$, or equivalently to the gradient $\mathbf{u}_{\mathbf{z}}$ of f at $\mathbf{z}$.

In this paper, we call points in $\mathbb{R}^m$ *states* and use them to represent finite sets of disks in $\mathbb{R}^2$. For $m = 3n$, the state $\mathbf{z}$ represents the set of disks $B_i = (z_i, r_i)$, for $0 \leq i \leq n-1$, where $[\mathbf{z}_{3i+1}, \mathbf{z}_{3i+2}]^T = z_i$ is the center and $\mathbf{z}_{3i+3} = r_i$ is the radius of B_i. The perimeter and area of the union of disks are maps $P, A : \mathbb{R}^{3n} \to \mathbb{R}$. Their derivatives at a state $\mathbf{z} \in \mathbb{R}^{3n}$ are linear maps $\mathrm{D}P_{\mathbf{z}}, \mathrm{D}A_{\mathbf{z}} : \mathbb{R}^{3n} \to \mathbb{R}$, and the goal of this paper is to give a complete description of these derivatives.

Voronoi decomposition. A basic tool in our study of derivatives is the Voronoi diagram, which decomposes the union of disks into convex cells. To describe it, we define the *power distance* of a point $x \in \mathbb{R}^2$ from B_i as $\pi_i(x) = \|x - z_i\|^2 - r_i^2$. The disk thus contains all points with non-positive power distance, and the bounding circle consists of all points with zero power distance from B_i. The *bisector* of two disks is the line of points with equal power distance to both. Given a finite collection of disks, the *(weighted) Voronoi cell* of B_i in this collection is the set of points x for which B_i minimizes the power distance,

$$V_i = \{x \in \mathbb{R}^3 \mid \pi_i(x) \leq \pi_j(x),\ \forall j\}.$$

Each Voronoi cell is the intersection of finitely many closed half-spaces and thus a convex polygon. The cells cover the entire plane and have pairwise disjoint interiors. The *(weighted) Voronoi diagram* consists of all Voronoi cells, their edges and their vertices. If we restrict the diagram to within the union of disks, we get a decomposition into convex cells. Figure 1 shows such a decomposition overlayed with the same after a small motion of the four disks. For the purpose of this paper, we may assume the disks are in general position, which implies that each Voronoi vertex belongs to exactly three Voronoi cells. The *Delaunay triangulation* is dual to the Voronoi diagram. It is obtained by taking the disk centers as vertices and drawing an edge and triangle between two and three vertices whose corresponding Voronoi cells have a non-empty common intersection. The *dual complex* K of the disks is defined by the same rule, except that the non-empty common intersections are demanded of the Voronoi cells clipped to within their corresponding disks. For an example see Figure 1, which shows the dual complex

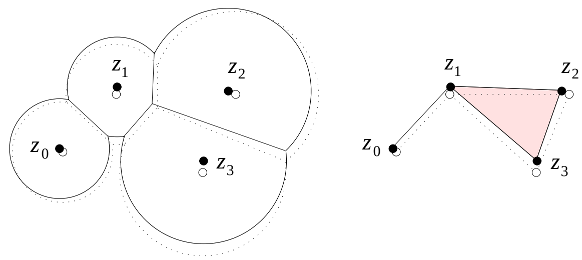

Fig. 1: Two snapshots of a decomposed union of four disks to the left and the dual complex to the right.

of four disks. The notion of neighborhood is formalized by defining the *link* of a vertex $z_i \in K$ as the set of vertices and edge connected to z_i by edges and triangles,

$$\mathrm{Lk}\, z_i = \{z_j, z_j z_k \mid z_i z_j, z_i z_j z_k \in K\}.$$

Similarly, the link of an edge $z_i z_j$ is the set of vertices connected to the edge by triangles, $\mathrm{Lk}\, z_i z_j = \{z_k \mid z_i z_j z_k \in K\}$. Since K is embedded in the plane, an edge belongs to at most two triangles which implies that the link contains at most two vertices.

Measuring. We use fractions to express the size of various geometric entities in the Voronoi decomposition. For example, β_i is the fraction of B_i contained in its Voronoi cell and σ_i is the fraction of the circle bounding B_i that belongs to the boundary of the union. The area and perimeter of the union are therefore

$$A \;=\; \pi \sum r_i^2 \beta_i \quad \text{and} \quad P \;=\; 2\pi \sum r_i \sigma_i.$$

We refer to the intersection points between circles as *corners*. The corner to the left of the directed line from z_i to z_j is x_{ij} and the one to the right is $\bar{x}_{ji}$. Note that $x_{ji} = \bar{x}_{ij}$. We use $\sigma_{ij} \in \{0, 1\}$ to indicate whether or not x_{ij} exists and lies on the boundary of the union. The total number of corners on the boundary is therefore $\sum \sigma_{ij}$. Finally, we define β_{ij} as the fraction of the chord $x_{ij} x_{ji}$ that belongs to the corresponding Voronoi edge.

Given the dual complex K of the disks, it is fairly straightforward to compute the β_i, σ_i, β_{ij}, and σ_{ij}. For example, $\sigma_{ij} = 1$ iff $z_i z_j$ is an edge in K and if $z_i z_j z_k$ is a triangle in K then z_k lies to the right of the directed line from z_i to z_j. We sketch inclusion-exclusion formulas for the remaining quantities. Proofs can be found in [6]. Define $B_i^j = \{x \in \mathbb{R}^2 \mid \pi_j(x) \leq \pi_i(x) \leq 0\}$, which is the portion of B_i of B_j's side of the bisector. Define $A_i^j = \mathrm{area}(B_i^j)$ and $A_i^{jk} = \mathrm{area}(B_i^j \cap B_i^k)$. Similarly, let P_i^j

and P_i^{jk} be the lengths of the circle arcs in the boundaries of B_i^j and $B_i^j \cap B_i^k$. Then

$$\beta_i = 1 - \left(\sum_j A_i^j - \sum_{j,k} A_i^{jk} \right) / \pi r_i^2,$$

$$\sigma_i = 1 - \left(\sum_j P_i^j - \sum_{j,k} P_i^{jk} \right) / 2\pi r_i,$$

where in both equations the first sum ranges over all vertices $z_j \in \mathrm{Lk}\, z_i$ and the second ranges over all $z_j z_k \in \mathrm{Lk}\, z_i$ in K. Finally, let $c_{ij} = x_{ij} x_{ji}$ be the chord defined by B_i and B_j and define $c_{ij}^k = \{x \in \mathbb{R}^2 \mid \pi_k(x) \leq \pi_i(x) = \pi_j(x) \leq 0\}$, which is the portion of c_{ij} on B_k's side of the bisectors. Define $r_{ij} = \mathrm{length}(c_{ij})/2$ and $L_{ij}^k = \mathrm{length}(c_{ij}^k)$. Then

$$\beta_{ij} = 1 - \left(\sum_k L_{ij}^k \right) / 2 r_{ij},$$

where the sum ranges are over all $z_k \in \mathrm{Lk}\, z_i z_j$ in K. The analytic formulas still required to compute the various areas and lengths can be found in [8], which also explains how the inclusion-exclusion formulas are implemented in the Alpha Shape software.

Motion. When we talk about a motion, we allow all $3n$ describing parameters to vary: each center can move in $\mathbb{R}^2$ and each radius can grow or shrink. When this happens, the union changes and so does the Voronoi diagram, as shown in Figure 1. In our approach to studying derivatives, we consider individual disks and look at how their Voronoi cells change. In other words, we keep a disk B_i fixed and study how the motion affects the portion of B_i that forms the cell in the clipped Voronoi diagram. This idea is illustrated in Figure 2. This approach suggests we understand the entire change as an accumulation

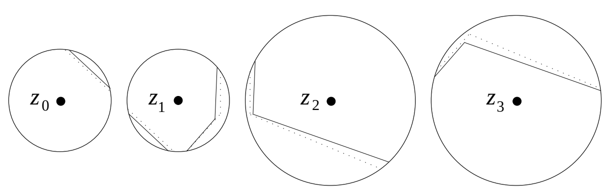

Fig. 2: Two snapshots of each disk clipped to within its Voronoi cell. The clipped disks are the same as in Figure 1, except that they are superimposed with fixed center.

of changes that happen to individual clipped disks, and we understand the change of an individual clipped disk as the accumulation of changes caused by neighbors in the Voronoi diagram. A central step in proving our results will therefore be the detailed analysis of the derivative in the interaction of two disks.

Theorems. The first result of this paper is a complete description of the derivative of the perimeter of a union of disks. Let $\zeta_{ij} = \|z_i - z_j\|$ be the distance between two centers. We write $u_{ij} = \frac{z_i - z_j}{\|z_i - z_j\|}$ for the unit vector between the same centers and v_{ij} for u_{ij} rotated through an angle of 90 degrees. Note that $u_{ji} = -u_{ij}$ and $v_{ji} = -v_{ij}$.

PERIMETER DERIVATIVE THEOREM. The derivative of the perimeter of a union of n disks with state $\mathbf{z} \in \mathbb{R}^{3n}$ is $\mathrm{D}P_{\mathbf{z}}(\mathbf{t}) = \langle \mathbf{p}, \mathbf{t} \rangle$, where

$$\begin{aligned}
\begin{bmatrix} \mathbf{p}_{3i+1} \\ \mathbf{p}_{3i+2} \end{bmatrix} &= \sum_{j \neq i} \left(p_{ij} \frac{\sigma_{ij} + \sigma_{ji}}{2} + q_{ij}(\sigma_{ij} - \sigma_{ji}) \right), \\
\mathbf{p}_{3i+3} &= 2\pi\sigma_i + \sum_{j \neq i} r_{ij} \frac{\sigma_{ij} + \sigma_{ji}}{2}, \\
p_{ij} &= \frac{r_i + r_j}{r_{ij}} \left(1 - \frac{(r_i - r_j)^2}{\zeta_{ij}^2} \right) \cdot u_{ij}, \\
q_{ij} &= \frac{r_j - r_i}{\zeta_{ij}} \cdot v_{ij}, \\
r_{ij} &= \frac{1}{r_{ij}} \left(\frac{(r_i - r_j)^2}{\zeta_{ij}} - \zeta_{ij} \right).
\end{aligned}$$

If $z_i z_j$ is not an edge in K then $\sigma_{ij} = \sigma_{ji} = 0$. We can therefore limit the sums in the Perimeter Derivative Theorem to all z_j in the link of z_i. If the link of z_i in K is a full circle then the perimeter and its derivative vanish. This is clear also from the formula because $\sigma_i = 0$ and $\frac{\sigma_{ij} + \sigma_{ji}}{2} = \sigma_{ij} - \sigma_{ji} = 0$ for all j.

The second result is a complete description of the derivative of the area of a disk union.

AREA DERIVATIVE THEOREM. The derivative of the area of a union of n disks with state $\mathbf{z}$ is $\mathrm{D}A_{\mathbf{z}}(\mathbf{t}) = \langle \mathbf{a}, \mathbf{t} \rangle$, where

$$\begin{aligned}
\begin{bmatrix} \mathbf{a}_{3i+1} \\ \mathbf{a}_{3i+2} \end{bmatrix} &= \sum_{j \neq i} a_{ij} \beta_{ij}, \\
\mathbf{a}_{3i+3} &= 2\pi r_i \sigma_i, \\
a_{ij} &= 2 r_{ij} \cdot u_{ij}.
\end{aligned}$$

We can again limit the sum to all z_j in the link of z_i. If the link of z_i in K is a full circle then the area derivative vanishes. Indeed, $\mathbf{a}_{3i+3} = 0$ because $\sigma_i = 0$ and $\sum a_{ij} \beta_{ij} = 0$ because of the Minkowski theorem for convex polygons.

3 Perimeter Derivative

In this section, we prove the Perimeter Derivative Theorem stated in Section 2. We begin by introducing some notation, continue by analyzing the cases of two and of n disks, and conclude by investigating when the derivative is not continuous.

Notation. For the case of two disks, we use the notation shown in Figure 3. The two disks are $B_0 = (z_0, r_0)$ and $B_1 = (z_1, r_1)$. We assume that the two bounding circles intersect in two corners, x and $\bar{x}$. Let r_{01} be half the distance between the two corners. Then $\zeta_0 = \sqrt{r_0^2 - r_{01}^2}$ is the distance between z_0 and the bisector, and similarly, ζ_1 is the distance between z_1 and the bisector. If z_0 and z_1 lie on different sides of the bisector then $\zeta = \zeta_0 + \zeta_1$ is the distance between the centers. We have $r_0^2 - r_1^2 = \zeta_0^2 - \zeta_1^2 = \zeta(\zeta_0 - \zeta_1)$ and therefore

$$\zeta_i = \frac{1}{2}\left(\zeta + \frac{r_i^2 - r_{1-i}^2}{\zeta}\right), \tag{1}$$

for $i = 0, 1$. If the two centers lie on the same side of the bisector, then $\zeta = \zeta_0 - \zeta_1$ is the distance between the centers. We have $r_0^2 - r_1^2 = \zeta_0^2 - \zeta_1^2 = \zeta(\zeta_0 + \zeta_1)$ and again Equation (1) for ζ_0 and ζ_1. Let θ_0 be the angle $\angle z_1 z_0 x$ at z_0, and similarly define

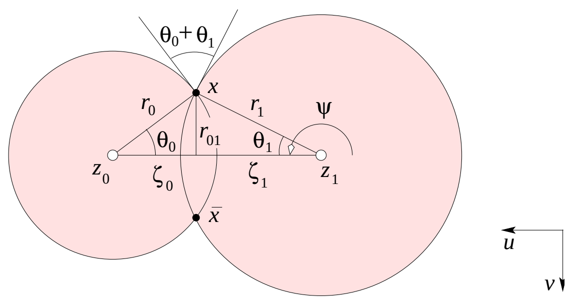

Fig. 3: Two disks bounded by intersecting circles and the various lengths and angles they define.

$\theta_1 = \angle z_0 z_1 \bar{x} = \angle x z_1 z_0$. Then

$$\theta_i = \arccos \frac{\zeta_i}{r_i}, \tag{2}$$

for $i = 0, 1$, and we note that $\theta_0 + \theta_1$ is the angle formed at the two corners. The contributions of each disk to the perimeter and the area of $B_0 \cup B_1$ are

$$P_i = 2(\pi - \theta_i) r_i, \tag{3}$$

$$A_i = (\pi - \theta_i) r_i^2 + r_{ij}\zeta_i, \tag{4}$$

for $i = 0, 1$. The perimeter of the union is $P = P_0 + P_1$, and the area is $A = A_0 + A_1$.

Motion. We study the derivative of P under motion by fixing z_1 and moving the other center along a smooth curve $\gamma(s)$, with $\gamma(0) = z_0$. At z_0 the velocity vector of the motion is $t = \frac{\partial \gamma}{\partial s}(0)$. Let $u = \frac{z_0 - z_1}{\|z_0 - z_1\|}$ and v be the unit vector obtained by rotating u through an angle of 90 degrees. We decompose the motion into a slope preserving and a distance preserving component, $t = \langle t, u\rangle u + \langle t, v\rangle v$. We compute the two partial derivatives with respect to the distance ζ and an angular motion. We use Equations (1), (2), and (3) for $i = 0$ to compute the derivative of P_0 with respect to the center distance,

$$\begin{aligned}\frac{\partial P_0}{\partial \zeta} &= \frac{\partial P_0}{\partial \theta_0} \cdot \frac{\partial \theta_0}{\partial \zeta_0} \cdot \frac{\partial \zeta_0}{\partial \zeta} \\ &= (-2r_0) \cdot \left(-\frac{1}{r_0 \sin \theta_0}\right) \cdot \left(\frac{1}{2} - \frac{r_0^2 - r_1^2}{2\zeta^2}\right) \\ &= \frac{r_0}{r_{01}} \left(1 - \frac{r_0^2 - r_1^2}{\zeta^2}\right).\end{aligned}$$

By symmetry, $\frac{\partial P_1}{\partial \zeta} = \frac{r_1}{r_{01}}(1 - \frac{r_1^2 - r_0^2}{\zeta^2})$. The derivative of P is the sum of the two derivatives, and therefore

$$\frac{\partial P}{\partial \zeta} = \frac{r_0 + r_1}{r_{01}} \left(1 - \frac{(r_0 - r_1)^2}{\zeta^2}\right). \tag{5}$$

To preserve distance we rotate z_0 around z_1 and let ψ denote the angle defined by the vector u. During the rotation the perimeter does of course not change. The reason is that we loose or gain the same amount of length at the leading corner, $\bar{x}$, as we gain or loose at the trailing corner, x. Since we have to deal with situations where one corner is exposed and the other is covered by other disks, we are interested in the derivative of the contribution near x, which we denote by $P_x(\psi)$. We have a gain on the boundary of B_1 minus a loss on the boundary of B_0, namely

$$\frac{\partial P_x}{\partial \psi} = \frac{r_1 - r_0}{\zeta}. \tag{6}$$

As mentioned above, the changes at the two corners cancel each other, or equivalently, $\frac{\partial P}{\partial \psi} = \frac{\partial P_x}{\partial \psi} + \frac{\partial P_{\bar{x}}}{\partial \psi} = \frac{r_1 - r_0}{\zeta} + \frac{r_0 - r_1}{\zeta} = 0$.

Growth. We grow or shrink the disk B_0 by changing its radius, r_0. Using Equations (1), (2), and (3) for $i = 0$ as before, we get

$$\begin{aligned}\frac{\partial P_0}{\partial r_0} &= \frac{\partial P_0}{\partial \theta_0} \cdot \frac{\partial \theta_0}{\partial \zeta_0} \cdot \frac{\partial \zeta_0}{\partial r_0} \\ &= \left(2(\pi - \theta_0)\frac{\partial \zeta_0}{\partial \theta_0}\frac{\zeta}{r_0} - 2r_0\right) \cdot \frac{\partial \theta_0}{\partial \zeta_0} \cdot \frac{r_0}{\zeta} \\ &= 2(\pi - \theta_0) + \frac{2(r_0^2 - \zeta\zeta_0)}{r_{01}\zeta}\end{aligned}$$

because $\frac{\partial \theta_0}{\partial \zeta_0} = -\frac{1}{\sin \theta_0}\frac{r_0^2 - \zeta\zeta_0}{r_0^3}$ and $\frac{1}{\sin \theta_0} = \frac{r_0}{r_{01}}$. The computation of the derivative of P_1 is more straightforward because r_1 and ζ both remain constant as r_0 changes. Using Equations (1), (2), and (3) for $i = 1$, we get

$$\frac{\partial P_1}{\partial r_0} = \frac{\partial P_1}{\partial \theta_1} \cdot \frac{\partial \theta_1}{\partial \zeta_1} \cdot \frac{\partial \zeta_1}{\partial r_0} \quad = \quad (-2r_1) \cdot \left(-\frac{1}{r_1 \sin \theta_1}\right) \cdot \left(-\frac{r_0}{\zeta}\right) \quad = \quad -\frac{2r_0 r_1}{r_{01}\zeta}.$$

Note that $\frac{2r_0^2-2r_0r_1}{\zeta}-\zeta_0$ is equal to $\frac{(r_0-r_1)^2}{\zeta}-\zeta$. The derivative of the perimeter, which is the sum of the two derivatives, is therefore

$$\frac{\partial P}{\partial r_0} = 2(\pi-\theta_0) + \frac{1}{r_{01}}\left(\frac{(r_0-r_1)^2}{\zeta}-\zeta\right). \tag{7}$$

The first term on the right in Equation (7) is the rate of growth if we scale the entire disk union. The second term accounts for the angle at which the two circles intersect. It is not difficult to show that this term is equal to $-2\cos(\theta_0+\theta_1)-\frac{2}{\sin(\theta_0+\theta_1)}$, which is geometrically the obvious dependence of the derivative on the angle between the two circles, as can be seen in Figure 3.

Assembly of relations. Let P be the perimeter of the union of disks B_i, for $0 \leq i \leq n-1$. By linearity, we can decompose the derivative along a curve with velocity vector $\mathbf{t} \in \mathbb{R}^{3n}$ into components. The i-th triplet of coordinates describes the change for B_i. The first two of the three coordinates give the velocity vector t_i of the center z_i. For each other disk B_j, we decompose that vector into a slope and a distance preserving component, $t_i = \langle t_i, u_{ij}\rangle u_{ij} + \langle t_i, v_{ij}\rangle v_{ij}$.

The derivative of the perimeter along the slope preserving direction is given by Equation (5). The length of the corresponding vector p_{ij} in the theorem is this derivative times the fractional number of boundary corners defined by B_i and B_j, which is $\frac{\sigma_{ij}+\sigma_{ji}}{2}$. The derivative along the distance preserving direction is given by Equation (6). The length of the corresponding vector q_{ij} in the theorem is that derivative times $\sigma_{ij}-\sigma_{ji}$, since we gain perimeter at the corner x_{ij} and loose at x_{ji} (or vice versa, if $\langle t_i, v_{ij}\rangle < 0$). The derivative with respect to the radius is given in Equation (7). The first term of that equation is the angle of B_i's contribution to the perimeter, which in the case of n disks is $2\pi\sigma_i$. The second term accounts for the angles at the two corners. It contributes to the derivative only for corners that belong to the boundary of the union. We thus multiply the corresponding term r_{ij} in the theorem by the fractional number of boundary corners. This completes the proof of the Perimeter Derivative Theorem.

4 Area Derivative

In this section, we prove the Area Derivative Theorem stated in Section 2. We use the same notation as in Section 3, which is illustrated in Figure 3.

Motion. As before we consider two disks $B_0 = (z_0, r_0)$ and $B_1 = (z_1, r_1)$, we keep z_1 fixed, and we move z_0 along a curve with velocity vector t at z_0. The unit vectors u and v are defined as before, and the motion is again decomposed into a slope and a distance preserving component, $t = \langle t, u\rangle u + \langle t, v\rangle v$. The distance preserving component does not change the area and has zero contribution to the area derivative. To compute the derivative with respect to the slope preserving motion, we use Equations (2) and (4) for $i = 0$ to get the derivative of A_0 with respect to ζ_0,

$$\frac{\partial A_0}{\partial \zeta_0} = -r_0^2\frac{\partial \theta_0}{\partial \zeta_0} + \frac{\partial r_{01}}{\partial \zeta_0}\zeta_0 + r_{01} \;=\; \frac{r_0^2}{r_{01}} - \frac{\zeta_0^2}{r_{01}} + r_{01} \;=\; 2r_{01}.$$

Symmetrically, we get $\frac{\partial A_1}{\partial \zeta_1} = 2r_{01}$. The derivative of the area with respect to the distance between the centers is $\frac{\partial A_0}{\partial \zeta_0} \cdot \frac{\partial \zeta_0}{\partial \zeta} + \frac{\partial A_1}{\partial \zeta_1} \cdot \frac{\partial \zeta_1}{\partial \zeta}$, which is

$$\frac{\partial A}{\partial \zeta} = 2r_{01}, \tag{8}$$

because $\partial\zeta_0 + \partial\zeta_1 = \partial\zeta$. This result is obvious geometrically, because to the first order the area gained is the rectangle with width $\partial\zeta$ and height $2r_{01}$ obtained by thickening the portion of the separating Voronoi edge.

Growth. Using Equation (4) for $i = 0$ and 1, we get

$$\begin{aligned}\frac{\partial A}{\partial r_0} &= \frac{\partial A_0}{\partial r_0} + \frac{\partial A_1}{\partial r_0} \\ &= 2(\pi - \theta_0)r_0 - \left(r_0^2 \frac{\partial \theta_0}{\partial r_0} + r_1^2 \frac{\partial \theta_1}{\partial r_0} \right) \\ &\quad + \frac{\partial r_{01}}{\partial r_0}(\zeta_0 + \zeta_1) + r_{01} \left(\frac{\partial \zeta_0}{\partial r_0} + \frac{\partial \zeta_1}{\partial r_0} \right).\end{aligned}$$

The right hand side consists of four terms of which the fourth vanishes because $\partial\zeta_0 + \partial\zeta_1 = 0$. The third term equals $\frac{r_0}{r_{01}\zeta}\zeta_1\zeta$. The second term is $r_0^2(\frac{\zeta_0\zeta - r_0^2}{r_{01}r_0^2})\frac{r_0}{\zeta} + r_1^2\frac{r_0}{r_{01}\zeta}$. The second and third terms cancel each other because $r_0^2 - r_1^2 + \zeta_1\zeta - \zeta_0\zeta = 0$. Hence,

$$\frac{\partial A}{\partial r_0} = 2(\pi - \theta_0)r_0. \tag{9}$$

This equation is again obvious geometrically because to the first order the gained area is the fraction of the annulus of width ∂r_0 and length $P_0 = 2(\pi - \theta_0)r_0$ obtained by thickening the boundary arc contributed by B_0.

Assembly of relations. Let A be the area of the union of disks B_i, for $0 \leq i \leq n-1$. We decompose the derivative into terms, as before. The derivative along the slope preserving direction is given by Equation (8). The length of the corresponding vector a_{ij} in the theorem is this derivative times the fractional chord length, which is β_{ij}. The derivative with respect to the radius is given by Equation (9). It is equal to the contribution of B_i to the perimeter, which in the case of n disks is $2\pi r_i \sigma_i$. This completes the proof of the Area Derivative Theorem.

5 Discussion

Consider a finite collection of disks in the plane. We call a motion that does not change radii and that at no time decreases the distance between any two centers a *continuous expansion*. The Area Derivative Theorem implies that the derivative along a continuous expansion is always non-negative. The area is therefore monotonously non-decreasing. This is not new and has been proved for general dimensions in 1998 by

Csikós [5]. The more restricted version of this result for unit-disks in the plane has been known since 1968. Bollobás' proof uses the fact that for unit disks the perimeter is also monotonously non-decreasing along continuous expansions [3]. Perhaps surprisingly, this is not true if the disks in the collection have different radii. The critical term that spoils the monotonicity is contributed by the rotational motion of one disk about another. That contribution can be non-zero if exactly one of the two corners defined by the two circles belongs to the boundary of the union. Continuous expansions that decrease the perimeter are therefore possible, and one is shown in Figure 4.

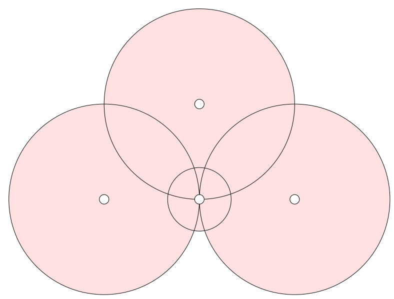

Fig. 4: Moving the small disk vertically downward does not decrease any distances but does decrease the perimeter.

References

1. M. P. BENDSØE. *Optimization of Structural Topology, Shape, and Material.* Springer-Verlag, Berlin, Germany, 1995.
2. M. P. BENDSØE AND C. A. MOTA SOARES (EDS.) *Topology Design of Structures.* Kluewer, Dordrecht, The Netherlands, 1993.
3. B. BOLLOBÁS. Area of the union of disks. *Elem. Math.* **23** (1968), 60–61.
4. R. BRYANT, H. EDELSBRUNNER, P. KOEHL AND M. LEVITT. The area derivative of a space-filling diagram. Manuscript, 2001.
5. B. CSIKÓS. On the volume of the union of balls. *Discrete Comput. Geom.* **20** (1998), 449–461.
6. H. EDELSBRUNNER. The union of balls and its dual shape. *Discrete Comput. Geom.* **13** (1995), 415–440.
7. H. EDELSBRUNNER. Deformable smooth surface design. *Discrete Comput. Geom.* **21** (1999), 87–115.
8. H. EDELSBRUNNER AND P. FU. Measuring space filling diagrams and voids. Rept. UIUC-BI-MB-94-01, Beckman Inst., Univ. Illinois, Urbana, Illinois, 1994.
9. M. LEVITT AND A. WARSHEL. Computer simulation of protein folding. *Nature* **253** (1975), 694–698.
10. M. SPIVAK. *Calculus on Manifolds.* Addison-Wesley, Reading, Massachusetts, 1965.