Inequalities for the Curvature of Curves and Surfaces *

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ABSTRACT

In this paper, we bound the difference between the total mean curvatures of two closed surfaces in $\mathbb{R}^3$ in terms of their total absolute curvatures and the Fréchet distance between the volumes they enclose. The proof relies on a combination of methods from algebraic topology and integral geometry. We also bound the difference between the lengths of two curves using the same methods.

Categories and Subject Descriptors

F.2.2 [**Analysis of Algorithms and Problem Complexity**]: Nonnumerical Algorithms and Problems—*Geometrical problems and computations, Computations on discrete structures*; G.2.1 [**Discrete Mathematics**]: Combinatorics—*Counting problems*

General Terms

Algorithms, Theory

Keywords

Integral geometry, curvature, Fréchet distance, persistence diagrams, bottleneck distance, approximation, stability.

1. INTRODUCTION

Given an oriented hypersurface S smoothly embedded in $\mathbb{R}^d$, the *mean curvature* of S at $p \in S$ is the average of the $d-1$ principal curvatures at p. The *total mean curvature* is the integral of the mean curvature over S. It is the second in a sequence of d quantities associated with a hypersurface, called *Quermassintegrale*, *Minkowski functionals*, or *Lipschitz-Killing curvatures* in the literature [3]. The total mean curvature and its relatives have many interesting properties and have been extensively studied by mathematicians during the last 150 years.

While curvature is *a priori* defined only for smooth objects, total mean curvature can be defined consistently for a large class of non-smooth, including piecewise-linear objects [15]. This is of practical interest since smooth surfaces are often represented by approximating meshes. From this perspective, an important question is whether the total mean curvature of a piecewise-linear surface is close to the one of approximated the smooth surface. More generally, understanding the behavior of Lipschitz-Killing curvatures under approximations is an important unsolved theoretical question [12]. In this direction, a convergence result for the Lipschitz-Killing curvatures of increasingly fine triangulations inscribed in a smooth submanifold was obtained in [11]. For surfaces, this result was strengthened in [5, 7] by extending it to curvature tensors and by giving an explicit error bound. An essential requirement in the latter approximation result is that the normals to the mesh are close to the normals to the smooth surface.

In this paper, we give an approximation result for the total mean curvature of surfaces under weaker conditions, using a stability result on topological persistence proved in [6]. We show that controlling the Fréchet distance between the smooth object and its approximation is sufficient to guarantee that their total mean curvatures are close, provided the approximation has bounded total absolute curvature. Surprisingly, the closeness between the normals to the two surfaces is not explicitly required. We prove this using the integral-geometric interpretations of the total absolute and the total mean curvatures [14]. Our method also yields a simple and apparently new geometric inequality for curves. More precisely, we obtain a bound on the difference between the lengths of two curves as a function of their Fréchet distance and of their total curvature. This result can be viewed as a generalization of Fáry's theorem [10].

Outline. Section 2 recalls the necessary background on topological persistence and integral geometry. Section 3 states and proves our results on surfaces and curves. Section 4 concludes the paper.

2. BACKGROUND

Before we state and prove our results, we review the necessary background. We need to understand topological persistence, a topic in algebraic topology [13], and we need integral-geometric interpretations of curvature and length, concepts studied in differential geometry [8].

Topological persistence. Given a generic smooth function f on a manifold M, Morse theory shows that the changes in the topology of the sublevel sets $f^{-1}(-\infty, x]$ are related to the critical points of f. More precisely, when the level x increases and passes a critical

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value, a handle is attached to the current sublevel set. This operation can have two kinds of consequences: either a new homology class is created or some homology class is destroyed. Topological persistence is a canonical way to pair up critical values that create homology classes with critical values that destroy them [9]. Each such pair forms a half-open *persistence interval* $[a_1, a_2)$ that can be interpreted as the life-span of a topological feature in the filtration of M by sublevel sets. An example is shown in Figure 1. Sweeping the curve from bottom to top, we encounter five critical

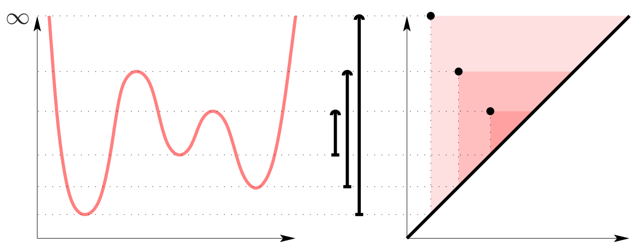

Figure 1: A one-dimensional real-valued function, its persistence intervals, and its persistence diagram. The dark, medium, light regions corresponds to persistent Betti numbers 3, 2, 1.

values in sequence. The first three create a new component each. The fourth critical value merges the component created last with the component created just before. The life-span of the third component is thus the interval between the third critical value and the fourth, as indicated in the middle of the figure. Similarly, the fifth critical value merges the component created by the second critical value with the one created by the first, and is therefore paired with the second value. The component created first is never destroyed and its life-span is the half-infinite interval starting at the first critical value. Because the function in Figure 1 is one-dimensional, the only interesting features in the sublevel sets are their connected components, which correspond to classes in the 0-th homology group, but persistence intervals can be defined similarly for k-th homology groups, for arbitrary k. Also, they can be defined for a large class of functions, including height functions on simplicial complexes in addition to Morse functions on smooth manifolds [6]. In this more general setting, critical values are replaced by *homological critical values* that correspond to levels at which the homology groups of the sublevel sets change.

Persistence diagrams. As shown in [4], the (multi-) set of all persistence intervals of a function provides a rather complete description of the topological relationships between different sublevel sets. For example, the image of the map from $\mathsf{H}_k(f^{-1}(-\infty, x])$ to $\mathsf{H}_k(f^{-1}(-\infty, y])$ induced by inclusion on k-th homology groups has dimension equal to the number of persistence intervals that contain both x and y. These numbers are called *persistent Betti numbers* and we denote them by $\beta_k(x, y)$. To visualize them, we represent each persistence interval $[a_1, a_2)$ by the point (a_1, a_2) in the plane, allowing infinite coordinates for unbounded intervals. The (multi-) set of all such points, together with the diagonal, $\Delta = \{(x, x) \mid x \in \mathbb{R}\}$, is the *persistence diagram* of the function, which we illustrate in Figure 1 on the right. For a given function f, there is of course one persistence diagram for each dimension k, denoted by $\mathrm{D}_k(f)$. We can now reformulate the above property of persistent Betti numbers.

k-TRIANGLE LEMMA [9]. The persistent Betti number $\beta_k(x, y)$ is the total number of points in $\mathrm{D}_k(f)$ contained in the quadrant $[-\infty, x] \times (y, +\infty]$.

Another important property of topological persistence is its stability under perturbation. We measure the distance between points in the extended plane, $\bar{\mathbb{R}}^2 = [-\infty, \infty]^2$, as $\|a - a'\|_\infty = \max\{|a_1 - a'_1|, |a_2 - a'_2|\}$. Given two multisets of points A and A' in $\bar{\mathbb{R}}^2$, we define their *bottleneck distance* as

$$d_B(A, A') \quad = \quad \inf_\gamma \sup_a \|a - \gamma(a)\|_\infty,$$

where a ranges over all points of A and γ ranges over all bijections from A to A'.

STABILITY THEOREM [6]. If f and g are sufficiently regular real-valued functions defined over the same space, then

$$d_B(\mathrm{D}_k(f), \mathrm{D}_k(g)) \le \sup_p |f(p) - g(p)|,$$

for all $k \ge 0$.

In other words, persistence diagrams are stable under possibly irregular perturbations of small amplitude, as illustrated in Figure 2.

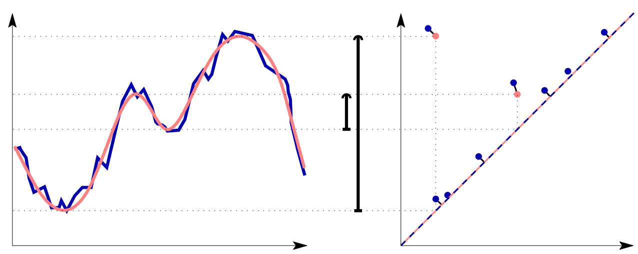

Figure 2: Left: two close functions, one with many and the other with just four critical values. Right: the persistence diagrams of the two functions, and the bijection between them.

Integral geometry for surfaces. Given a smoothly embedded surface S in $\mathbb{R}^3$, we recall that the two principal curvatures at a point $p \in S$ are the maximum and minimum normal curvatures, denoted as $\kappa_1(p) \ge \kappa_2(p)$. The *total mean curvature* is the integral of the mean curvature and the *total absolute curvature* is the integral of the absolute Gaussian curvature:

$$H(S) \quad = \quad \frac{1}{2} \int_S (\kappa_1(p) + \kappa_2(p)) \, \mathrm{d}p,$$

$$G(S) \quad = \quad \int_S |\kappa_1(p)\kappa_2(p)| \, \mathrm{d}p.$$

The latter is also the total area swept by the normal vector and measures the intuitive notion of bumpiness. For example, surfaces with zero total absolute curvature are developable, that is, they are everywhere locally isometric to the plane. To prove our results, we will use the integral-geometric interpretation of total absolute and total mean curvature, which we now recall. We assume S is a compact surface without boundary.

TOTAL ABSOLUTE CURVATURE INTERPRETATION [14]. The total absolute curvature, $G(S)$, equals 2π times the average number of critical points of a height function on the surface, where the direction is chosen uniformly at random from $\mathbb{S}^2$.

One advantage of this interpretation is that it generalizes to piecewise-linear surfaces, for which there is a well-defined notion of critical points for height functions [2]. These points are considered with multiplicity, counting an i-fold saddles as i critical points. It is not difficult to derive an elementary sum for the total absolute curvature of a piecewise-linear surface in terms of face angles [17]. To state the integral-geometric interpretation of the total mean curvature, we introduce the compact body $\bar{S} \subseteq \mathbb{R}^3$ with boundary S.

TOTAL MEAN CURVATURE INTERPRETATION [14]. The total mean curvature, $H(S)$, equals the integral, over all planes, of the Euler characteristic of the intersection of the plane with $\bar{S}$.

For this interpretation to hold, we need a measure on the set of planes that is invariant under rigid motions. This measure can be defined by parametrizing the planes by the direction of the unit normal and the distance from the origin. Using the same letter for a unit vector, $u \in \mathbb{S}^2$, and the dot product with that vector, $u : \mathbb{R}^3 \to \mathbb{R}$, we have

$$H(S) \;=\; \frac{1}{2} \int_{\mathbb{S}^2} \int_{-\infty}^{+\infty} \chi(\bar{S} \cap u^{-1}(z)) \, \mathrm{d}z \mathrm{d}u.$$

Again, this expression of total mean curvature extends directly to the piecewise-linear case. Computations show that for a piecewise-linear surface, the total mean curvature is a sum over all edges, in which the contribution of an edge is its length times the signed angle between the normals of the two incident triangles [7].

Integral geometry for curves. Similar formulas hold for piecewise-linear or smooth curves, with or without self-intersections. We write $C : \mathbb{S}^1 \to \mathbb{R}^n$ for a *closed curve* and $C : [0,1] \to \mathbb{R}^n$ for an *open curve*. Without causing any confusion, we will abuse notation and use C to denote both the map and its image, which is a geometric set. If C is smooth, we can define its *curvature*, $\kappa(s)$, at every point $p = C(s)$. Assuming a parametrization with constant speed $\sigma = \|\dot{C}(s)\|$, we have $\kappa(s) = \|\ddot{C}(s)\| / \sigma^2$. The *total curvature* is the distance traveled by the unit tangent vector or, equivalently, the integral $K(C) = \int \kappa(s) \, \mathrm{d}s$ over $\mathbb{S}^1$ or $[0,1]$.

TOTAL CURVATURE INTERPRETATION [14]. The total curvature, $K(C)$, of a closed curve equals π times the average number of critical points of $u \circ C : \mathbb{S}^1 \to \mathbb{R}$, where u is chosen uniformly at random from $\mathbb{S}^{n-1}$.

For open curves, we count only the interior critical points and not the two endpoints. In the case of a piecewise-linear curve, this integral-geometric quantity is still defined and coincides with what we would expect, namely the sum of all unsigned bend angles at the vertices. In lieu of the total mean curvature, we consider the integral-geometric interpretation of the length.

LENGTH INTERPRETATION [14]. The length, $L(C)$, of an open or closed curve equals π over the volume of $\mathbb{S}^n$ times the integral, over all hyperplanes, of the number of intersection points between the curve and the hyperplane.

Using again the parametrization of the hyperplanes by direction and distance from the origin, we can write this more formally using the Cauchy-Crofton formula:

$$L(C) \;=\; \frac{\pi}{\mathrm{vol}(\mathbb{S}^n)} \int_{\mathbb{S}^{n-1}} \int_{-\infty}^{+\infty} \sharp((u \circ C)^{-1}(z)) \, \mathrm{d}z \mathrm{d}u,$$

where $\sharp$ counts the intersections between C and the hyperplane $u^{-1}(z)$. By $\mathrm{vol}(\mathbb{S}^n)$ we mean of course the n-dimensional measure of the n-dimensional unit sphere.

3. RESULTS

We now combine the Stability Theorem for topological persistence with the integral-geometric interpretation of curvature and length to compare geometric quantities associated with curves and surfaces that are close under the Fréchet distance.

Surfaces. To state our result, we let S_1 and S_2 be two connected, compact surfaces in $\mathbb{R}^3$, either smooth or piecewise-linear, bounding homeomorphic subsets $\bar{S}_1$ and $\bar{S}_2$ of $\mathbb{R}^3$. Let g be the common genus of S_1 and S_2. By definition, the *Fréchet distance* between the two bodies is $d_F(\bar{S}_1, \bar{S}_2) = \inf_\phi \sup_{p \in \bar{S}_1} \|p - \phi(p)\|_2$, where ϕ ranges over all homeomorphisms from $\bar{S}_1$ to $\bar{S}_2$. Letting H_i and G_i denote the total mean and total absolute curvatures of S_i, the first result is an upper bound on the difference between the total mean curvatures in terms of the total absolute curvatures and the Fréchet distance.

TOTAL MEAN CURVATURE THEOREM. Given two connected, compact surfaces S_1 and S_2 in $\mathbb{R}^3$, we have

$$|H_1 - H_2| \;\le\; [G_1 + G_2 - 4\pi(1+g)] \, d_F(\bar{S}_1, \bar{S}_2).$$

PROOF. We prove the inequality in four steps. First, we transform the integral-geometric interpretation of the total mean curvature from level sets to sublevel sets of height functions. Second, we express the integral of Euler characteristics of sublevel sets in terms of points in persistence diagrams. Third, we use the Stability Theorem to bound the difference between the expressions for the two surfaces. Fourth, we integrate this difference over all directions $u \in \mathbb{S}^2$ to get the claimed inequality.

For the first step, let R be large enough such that S_1 and S_2 are contained in the ball of radius R centered at the origin. Denoting by $u_i : \bar{S}_i \to \mathbb{R}$ the restriction of u to $\bar{S}_i$, the Total Mean Curvature Interpretation can be written as

$$H_i \;=\; \frac{1}{2} \int_{\mathbb{S}^2} \int_{-R}^{R} \chi(u_i^{-1}(z)) \, \mathrm{d}z \mathrm{d}u. \tag{1}$$

If S_i is piecewise linear or u_i is a Morse function on S_i then $\chi(u_i^{-1}(-\infty, z]) + \chi(u_i^{-1}[z, \infty)) = \chi(u_i^{-1}(z)) + \chi(\bar{S}_i)$ by additivity of the Euler characteristic. Since almost all height functions are Morse, we can use this formula within the integral and get

$$\begin{aligned} H_i \;=\; & \int_{\mathbb{S}^2} \int_{-R}^{R} \chi(u_i^{-1}(-\infty, z]) \, \mathrm{d}z \mathrm{d}u \\ & - \frac{1}{2} \int_{\mathbb{S}^2} \int_{-R}^{R} \chi(\bar{S}_i) \, \mathrm{d}z \mathrm{d}u, \end{aligned}$$

because the integral of $\chi(u_i^{-1}(-\infty, z])$ is the same as that of $\chi(u_i^{-1}[z, \infty))$. We note that the second term is equal to $4\pi R \chi(\bar{S}_i)$.

For the second step, we choose a direction $u \in \mathbb{S}^2$ that satisfies the above condition for both surfaces, that is u_i is Morse if S_i is smooth and there is no restriction if S_i is piecewise-linear. We focus on the term $\chi_i(z) = \chi(u_i^{-1}(-\infty, z])$. By definition, $\chi_i(z)$ is the alternating sum of Betti numbers of $u_i^{-1}(-\infty, z]$. By the k-Triangle Lemma, the k-th Betti number is the total number of points of $\mathrm{D}_k(u_i)$ within the quadrant $[-\infty, z] \times (z, \infty]$. For any point $a = (x, y) \in \bar{\mathbb{R}}^2$, we write $\mathbf{1}_a : \mathbb{R} \to \mathbb{R}$ for the indicator function defined by $\mathbf{1}_a(z) = 1$ if $x \le z < y$ and $\mathbf{1}_a(z) = 0$, otherwise. Using this notation, we have $\beta_k(u_i^{-1}(-\infty, z]) = \sum_{a \in \mathrm{D}_k(u_i)} \mathbf{1}_a(z)$. Hence,

$$\chi_i(z) \;=\; \sum_{a \in \mathrm{D}(u^i)} (-1)^{k_a} \mathbf{1}_a(z),$$

where $\mathrm{D}(u_i)$ is the union of the $\mathrm{D}_k(u_i)$ and k_a is the integer such that $a \in \mathrm{D}_{k_a}(u_i)$.

In the third step, we compare the integrals over z of $\chi_1(z)$ and $\chi_2(z)$. Let $\varepsilon = d_F(\bar{S}_1, \bar{S}_2)$. By definition of Fréchet distance, for any $\delta > 0$, there is a homeomorphism $\phi : \bar{S}_1 \to \bar{S}_2$ that moves points by at most $\varepsilon + \delta$. Hence $\|u_1 - u_2 \circ \phi\|_\infty \leq \varepsilon + \delta$. Persistence diagrams are invariant under change of variables, that is, $\mathrm{D}_k(u_2 \circ \phi) = \mathrm{D}_k(u_2)$. Thus by the Stability Theorem, the bottleneck distance between $\mathrm{D}_k(u_1)$ and $\mathrm{D}_k(u_2)$ is at most $\varepsilon + \delta$. We thus get a bijection $\psi : \mathrm{D}(u_1) \to \mathrm{D}(u_2)$ that moves points by at most $\varepsilon + \delta$ in the L_∞ metric and such that $k_{\psi(a)} = k_a$. As a consequence, the difference between the integrals over z of the Euler characteristics can be written as

$$
\begin{aligned}
X &= \left|\int_{-R}^{R} (\chi_1(z) - \chi_2(z))\,\mathrm{d}z\right| \\
&= \left|\sum_{a \in \mathrm{D}(u_1)} (-1)^{k_a} \int_{-R}^{R} (\mathbf{1}_a(z) - \mathbf{1}_{\psi(a)}(z))\,\mathrm{d}z\right| \\
&\leq \sum_{a \in \mathrm{D}(u_1)} \int_{-\infty}^{\infty} |\mathbf{1}_a(z) - \mathbf{1}_{\psi(a)}(z)|\,\mathrm{d}z.
\end{aligned}
$$

Each term in the above sum is 0, if both a and $\psi(a)$ lie on the diagonal, at most $\varepsilon + \delta$, if a and $\psi(a)$ are both off-diagonal and have one infinite coordinate each, and at most $2(\varepsilon + \delta)$, if a or $\psi(a)$ or both are off-diagonal with two finite coordinates each. In other words, we pay $\varepsilon + \delta$ for every finite coordinate, except if two such coordinates belong to corresponding off-diagonal points, in which case we pay $\varepsilon + \delta$ for only one. The total number of finite coordinates of off-diagonal points is at most the number of homological critical values of u_1 and u_2, counted with multiplicity in the piecewise linear case. To get an upper bound on the sum, we assume every off-diagonal point with two finite coordinates is mapped to a diagonal point in the other diagram. The off-diagonal points with one infinite coordinate are necessarily mapped to each other. By the k-Triangle Lemma, their number is the sum of Betti numbers $\beta_k(x, x)$ of $\bar{S}_1$, for sufficiently large x, which is also the sum of Betti numbers of $\bar{S}_1$, namely $1 + g$. Denoting the number of critical points of u_i by c_i, we thus get $X \leq \varepsilon(c_1 + c_2 - 1 - g)$ since δ can be chosen arbitrarily small.

We are ready for the fourth and last step. Recalling the definition of X, we plug the inequality for X into Equation (1) and get

$$
|H_1 - H_2| \;\leq\; \varepsilon \int_{\mathbb{S}^2} (c_1 + c_2 - 1 - g)\,\mathrm{d}u.
$$

The right hand side is $4\pi\varepsilon$ times the expected number of homological critical values of a random height function of $\bar{S}_1$ plus the same for $\bar{S}_2$, minus $4\pi(1+g)\varepsilon$. But the expected number of homological critical values of a height function on $\bar{S}_i$ is half the expected number of homological critical values of the same height function restricted to the boundary, S_i. Indeed, each point x of S_i gives rise to a homological critical value of exactly one height function on $\bar{S}_i$, namely u_i, where u is the inward normal at x. But if one considers restrictions of height functions to S_i, then x is critical both in the direction u and the direction $-u$. As a consequence, $4\pi\varepsilon$ times the expected number of critical points of a random height function of $\bar{S}_i$ is εG_i, by the Total Absolute Curvature Interpretation. The claimed inequality follows. $\square$

A consequence of the Total Mean Curvature Theorem is that the total mean curvature of a smooth surface can be estimated from a piecewise-linear approximation, provided the approximation is not too bumpy. The advantage of this result over the one proved in [5] is that it does not involve a bound on the angle between the normals to the two surfaces. The closeness between the normals is only controlled indirectly, through the total absolute curvature of the piecewise-linear approximation. Indeed, an approximation with highly inaccurate normals is likely to have large total absolute curvature. However, this is not necessarily the case. For instance, applying a "stair-like" perturbation on part of the surface can change its normals by almost 90° without substantially increasing its total absolute curvature. We also see that the accuracy of the estimate can possibly be improved by reducing the total absolute curvature of the piecewise-linear approximation, for instance using the edge-flip algorithm described in [17].

Curves. Following the pattern of the proof of the Total Mean Curvature Theorem, we now prove a similar inequality for the length difference between two curves. Let $C_1, C_2 : \mathbb{S}^1 \to \mathbb{R}^n$ be two closed curves, with or without self-intersections, which may be smooth or piecewise linear. The Fréchet distance between them is $d_F(C_1, C_2) = \inf_\phi \sup_s \|C_1(s) - C_2(\phi(s))\|_2$, which agrees with the definition of Fréchet distance between curves given in [1], except that we do not assume that ϕ preserves the orientation of the circle. Observe that the Fréchet distance does not depend on the parameterizations of the curves. Writing $L_i = L(C_i)$ for the length and $K_i = K(C_i)$ for the total curvature of C_i, we can now state and prove the result.

LENGTH THEOREM. Given two closed curves C_1 and C_2 in $\mathbb{R}^n$, we have

$$
|L_1 - L_2| \;\leq\; \frac{2\mathrm{vol}(\mathbb{S}^{n-1})}{\mathrm{vol}(\mathbb{S}^n)} [K_1 + K_2 - 2\pi]\; d_F(C_1, C_2).
$$

PROOF. The proof follows the same four steps as the proof of the Total Mean Curvature Theorem. We will therefore take shortcuts and focus on the differences, which are few.

First, we transform the integral-geometric interpretation of the length from level sets to sublevel sets of height functions. Fixing $u \in \mathbb{S}^{n-1}$, we write $u_i = u \circ C_i$, for $i = 1, 2$. Almost all level sets of the form $u_i^{-1}(z)$ consist of an even number of points. These points decompose C_i into arcs, half of which belong to $u_i^{-1}(-\infty, z]$. Hence,

$$
\int_{-\infty}^{\infty} \sharp(u_i^{-1}(z))\,\mathrm{d}z \;=\; 2\int_{-\infty}^{\infty} \chi_i(z)\,\mathrm{d}z,
$$

where $\chi_i(z) = \chi(u_i^{-1}(-\infty, z])$.

Second, we recall that for curves, the Euler characteristic of a sublevel set, $\chi_i(z)$, is the number of components minus the number of loops. Equivalently, $\chi_i(z)$ is the number of points of $\mathrm{D}_0(u_i)$ within the quadrant $[-\infty, z] \times (z, \infty]$ minus the number of points of $\mathrm{D}_1(u_i)$ within the same quadrant. Using the indicator function $\mathbf{1}_a(z)$, we can rewrite this number as

$$
\chi_i(z) \;=\; \sum_{a \in \mathrm{D}_0(u_i)} \mathbf{1}_a(z) - \sum_{a \in \mathrm{D}_1(u_i)} \mathbf{1}_a(z).
$$

Third, we compare the integrals of $\chi_1(z)$ and $\chi_2(z)$. Let $\varepsilon = d_F(C_1, C_2)$ be the Fréchet distance and $\phi : \mathbb{S}^1 \to \mathbb{S}^1$ a homeomorphism such that the Euclidean distance between points $C_1(s)$ and $C_2(\phi(s))$ is at most $\varepsilon + \delta$. It follows that $\|u_1 - u_2 \circ \phi\|_\infty \leq \varepsilon + \delta$. The Stability Theorem implies that the bottleneck distance between $\mathrm{D}_0(u_1)$ and $\mathrm{D}_0(u_2)$ is at most $\varepsilon + \delta$, and the same is true for $\mathrm{D}_1(u_1)$ and $\mathrm{D}_1(u_2)$. Let $\psi : \mathrm{D}(u_1) \to \mathrm{D}(u_2)$ be the corresponding bijection between the unions of the diagrams. The difference between the integrals of the Euler characteristic is

$$X = \left| \int_{-\infty}^{\infty} \chi_1(z)\,\mathrm{d}z - \int_{-\infty}^{\infty} \chi_2(z)\,\mathrm{d}z \right|$$
$$\leq \sum_{a \in \mathrm{D}(u_1)} \int_{-\infty}^{\infty} |\mathbf{1}_a(z) - \mathbf{1}_{\psi(a)}(z)|\,\mathrm{d}z.$$

As before, this sum is at most $\varepsilon + \delta$ times half the number of finite coordinates of off-diagonal points that are mapped to each other plus the number of finite coordinates of off-diagonal points that are mapped to diagonal points in the other diagram. There are two points with one infinite coordinate each, one in $\mathrm{D}_0(u_1)$ and the other in $\mathrm{D}_1(u_1)$, and they both map to the corresponding points in the two diagrams of u_2. This implies $X \leq \varepsilon(c_1 + c_2 - 2)$.

Fourth, we plug the bound for X into the Cauchy-Crofton formula and get

$$|L_1 - L_2| \leq \frac{2\varepsilon\pi}{\mathrm{vol}(\mathbb{S}^n)} \int_{\mathbb{S}^{n-1}} (c_1 + c_2 - 2)\,\mathrm{d}u.$$

The right hand side is $2\varepsilon\pi\mathrm{vol}(\mathbb{S}^{n-1})/\mathrm{vol}(\mathbb{S}^n)$ times the expected number of critical point of a random height function of C_1 plus the same for C_2 minus 2. Using the Total Curvature Interpretation, this is at most $2\varepsilon\mathrm{vol}(\mathbb{S}^{n-1})/\mathrm{vol}(\mathbb{S}^n)$ times $K_1 + K_2 - 2\pi$, as claimed. $\square$

The Length Theorem can be viewed as a generalization of Fáry's theorem [10], which states that the length of a closed curve contained in the unit disk cannot exceed its total curvature; see Figure 3. Indeed, if C_1 is such a curve and C_2 is a circle of radius δ

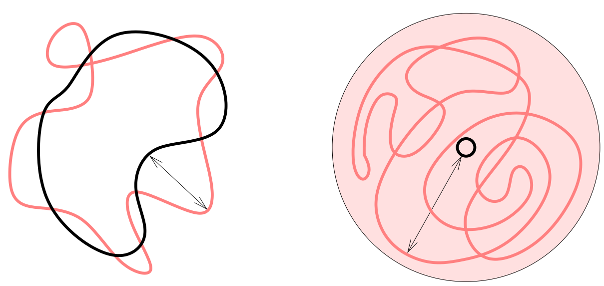

Figure 3: Left: our result compares the lengths of two curves as a function of their Fréchet distance indicated by the arrow. Right: Fáry's theorem can be obtained from ours by taking one of the curves to be a small circle centered at the origin.

centered at the origin, then the total curvature of C_2 is 2π and the Fréchet distance between C_1 and C_2 is at most $1+\delta$. Applying our result and letting δ go to zero gives the desired statement. Fáry's theorem has very short proofs [16], but none of them seem to extend to our more general case. On the other hand, Fáry's theorem holds in higher dimensions, but our methods only yields a weaker result in dimension higher than two. More precisely, we get that the length of a closed curve in the unit ball in $\mathbb{R}^n$ does not exceed 2vol $(\mathbb{S}^{n-1})$/vol $(\mathbb{S}^n)$ times its total curvature. The area of $\mathbb{S}^2$ is 4π and the volume of $\mathbb{S}^3$ is $2\pi^2$ implying that in $\mathbb{R}^3$ we get an extra factor of $4/\pi$. It would be interesting to know whether these constants are tight when the two curves are chosen without restriction in our theorem.

The proof of the Length Theorem can be extended from closed to open curves, for which we get

$$|L_1 - L_2| \leq \frac{2\mathrm{vol}(\mathbb{S}^{n-1})}{\mathrm{vol}(\mathbb{S}^n)} [K_1 + K_2 + \pi]\, d_F(C_1, C_2).$$

Alternatively, we can prove this inequality by reduction to the closed curve case. Double C_i by adding a slightly shifted copy of itself and connecting corresponding ends by sharp turns. The new length is $2L_i + \delta$, where $\delta > 0$ is arbitrarily small. The new total curvature is $2K_i + 2\pi$, and the Fréchet distance between the doubled curves is at most that between the original curves plus an arbitrarily small, positive amount. We get that $|2L_1 - 2L_2|$ is at most $2\mathrm{vol}(\mathbb{S}^{n-1})/\mathrm{vol}(\mathbb{S}^n)$ times $[2K_1 + 2K_2 + 2\pi]\ d_F(C_1, C_2)$ from the Length Theorem. Dividing by two gives the claimed inequality.

4. DISCUSSION

In this paper, we derived simple geometric inequalities for the curvature of curves and surfaces as consequences of the Stability Theorem for topological persistence. Several questions remain unanswered. First, is there a way to extend our result to the comparison of integrals of mean curvature on small regions of surfaces, as in [7, 11]? Second, does a result similar to ours hold for the integral of curvature tensors, as introduced in [7]? Finally, our result extends directly to the comparison of total mean curvatures of hypersurfaces in higher dimension. But what about other Lipschitz-Killing curvatures?

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