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Open Problems in Discrete and Computational Geometry

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We have selected problems that may not yet be well known, but have the potential to push the research in interesting directions. In particular, we state problems that do not require specific knowledge outside the standard circle of ideas in discrete geometry. Despite the relatively simple statements, these problems are related to current research and their solutions are likely to require new ideas and approaches. We have chosen problems from different fields to make this short paper attractive to a wide range of specialists.

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1. Random Persistence Diagrams

Random functions. Let $f : \mathbb{R}^d \rightarrow \mathbb{R}$ be a random continuous function of some sort. For example, f may be a Gaussian random field, as studied in [1]. It is convenient to compactify the domain, which we do by defining f with period 1.0 in every coordinate direction. In other words, \mathbb{X} is the d -dimensional unit cube with opposite faces identified, and $f : \mathbb{X} \rightarrow \mathbb{R}$ is a random continuous function on this d -torus. Another natural choice would be to pick n random points in \mathbb{X} and to define $f(x)$ equal to the Euclidean distance of x from the nearest of these random points. For each $\alpha \in \mathbb{R}$, let $\mathbb{X}^\alpha = f^{-1}[\alpha, \infty)$ be the *superlevel set* at α .

Persistent homology. We consider the persistence diagram of f [2], which we now sketch. Applying the homology functor to the increasing sequence of superlevel sets, we get a *tower* of homology groups, connected by homomorphisms induced by inclusion. Assuming f is tame, we have only finitely many different groups, all of finite rank.

Let α_i be the i -th homological critical value, and write V_i for the homology groups of \mathbb{X}^{α_i} . Assuming coefficients in a field, all homology groups are vector spaces, and all homomorphisms are linear maps, which we write as $f_{i,j} : V_i \rightarrow V_j$. We now define the *persistent homology groups*, $V_{i,j} = \text{im } f_{i,j}$, and the *persistent Betti numbers*, $\beta_{i,j} = \text{rank } V_{i,j}$. Finally, we define the multiset of *birth-death pairs*, (α_k, α_ℓ) , such that

$$\beta_{i,j} = \#\{(\alpha_k, \alpha_\ell) \mid \alpha_k \leq \alpha_i \text{ and } \alpha_j < \alpha_\ell\},$$

for all $i < j$. The multiset of birth-death pairs is unique and characterizes the tower up to isomorphisms. We represent each birth-death pair by the point $(\alpha_k + \alpha_\ell, \alpha_\ell - \alpha_k)$ in the plane, referring to the multiset of points as the *persistence diagram* of f . The vertical coordinate of a point is its *persistence*, and half of its horizontal coordinate is its *mean age*.

Intensity plots. Here, we are interested in the typical shape of a persistence diagram, assuming f is chosen at random from some population. To this end, we construct the *empirical intensity plot* of f , which is the function $p : \mathbb{R}^2 \rightarrow \mathbb{R}$ whose integral over every region $R \subseteq \mathbb{R}^2$ is the expected number of points in R . At this time, we have only empirical evidence that these intensity plots are well defined, at least in some cases; see Figure 1. It would be interesting to determine precise conditions on the random experiment under which the plots exist. Note that the plots in Figure 1 are symmetric, reflecting the symmetries of Gaussian random fields and Alexander duality.

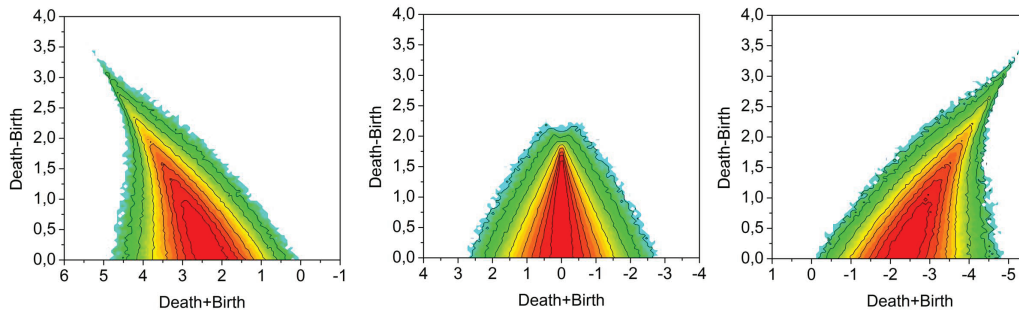


Рис. 1. The empirical intensity plots computed by averaging the persistence diagrams of ten Gaussian random fields on the 3-torus. From left to right: the plots for 0-, 1-, 2-dimensional homology. (Pictures courtesy of Pratyush Pranav).

Problem 1. We are interested in analytic descriptions of the intensity plots.

These will be difficult to get because even easier problems are still open today. On the positive side, we know the expectation of the Euler characteristic of the superlevel set as a function of α , and there two proofs, one using the Gauss-Bonnet Theorem [3] and the other the locality of critical points [1]. However, we do not know the analytic expression of the expected Betti numbers. More precisely, the behavior of the expected Betti numbers are well understood for small and for large values of α , but even β_0 is not well understood for values of α near the threshold at which the superlevel set gets connected. One can always hope that a more general approach gives the necessary structure to gain insight into this question. In this context, we note the “pointy hat” shape of the 0-dimensional plot, with its tip leaning to the left. This means that high persistence components tend toward higher mean age. Why?

2. Which Convex Bodies are Most Chiral?

Chirality from mirror images. The quantitative study of the symmetry of convex bodies has a long tradition in geometry [6]. Here, we follow a suggestion of Buda, auf der Heyde, Mislaw and quantify the asymmetry of a convex body using intersections with its mirror images [4]. Let B be a compact convex body with non-empty interior in \mathbb{R}^n . Given a hyperplane, ϱ , we write $B' = \varrho(B)$ for the mirror image obtained by reflecting B across ϱ . We are interested in the ratio of the n -dimensional volume of the intersection of B and B' over the volume of B . For symmetric bodies, there exist hyperplanes for which this ratio is one, and for others it is always less than one. To obtain a measure of how far the body is from being symmetric, we take the supremum over all hyperplanes and define

$$\chi(B) = 1 - \sup_{\varrho} \frac{\text{vol}(B \cap \varrho(B))}{\text{vol}(B)}.$$

This measure of chirality is a number in $[0, 1]$. It is zero for symmetric bodies, but it is not clear how close to one it can be. Note that we may alternatively consider all rigid

motions of the mirror image. Taking the supremum, we define

$$\chi^*(B) = 1 - \sup_{\mu} \frac{\text{vol}(B \cap \mu(B'))}{\text{vol}(B)},$$

where B' is a mirror image of B and $\mu(B')$ is its image under a rigid motion. Every reflection can be obtained by composing a fixed reflection and a rigid motion, but the converse is not true, which implies $\chi^*(B) \leq \chi(B)$.

Problem statement. There are a number of questions one can ask, about the computational complexity of computing χ and χ^* , about extremal properties of these measures, and more. Keeping in mind that the motivation for the question comes from chemistry, the most important case is $n = 3$.

Problem 2. What is the infimum of χ over all compact convex bodies with non-empty volume in \mathbb{R}^n ? If this infimum is attained, what is the solid body that attains it?

We can of course ask the same question for χ^* . It is not difficult to prove that $\chi(B) = \chi^*(B)$ in two dimensions; see Buda and Mislow [5]. Is this also true in three or higher dimensions? As an indication that the answer might be negative, we mention an example of G. Chelnokov which shows that there are bodies for which the two measures of chirality are different.

Other measures. Beyond the specific quantification of asymmetry using the volume of the intersection with a mirror image, we may ask for other measures.

Problem 3. What can be said about functions on the family of compact convex polytopes in \mathbb{R}^n that are easy to calculate, have reasonable continuity properties, are invariant under rigid motions but *not* under mirror imaging?

As a possible approach one may consider a more general object: a collection of points, p_i , and weights, w_i . Assuming that the mass center is at the origin, we may write the m -th moment as

$$M_m = \sum_i w_i \underbrace{p_i \otimes \cdots \otimes p_i}_m.$$

In other words, this is a homogeneous polynomial of degree m given by the formula $M_m(x) = \sum_i w_i \langle p_i, x \rangle^m$. Formally, for a homogeneous polynomial h , we want to quantify the condition that the $\text{SO}(3)$ -orbits of $h(x, y, z)$ and its mirror image, $h(-x, y, z)$, are distinct. It is clear, that for degrees 1 and 2 the polynomials $h(x, y, z)$ and $h(-x, y, z)$ are always within the same $\text{SO}(3)$ -orbit, but for $m \geq 3$ this may be not so; see [7]. We may now interpret the above question as a request to write down $\text{SO}(3)$ -invariant functions of homogeneous degree m polynomials allowing to detect polynomials that cannot be transformed by $\text{SO}(3)$ into their mirror images.

3. Random Triangles in the Plane

Consider a triplet of independent random points, x_0, x_1, x_2 , in the plane. By a *random point* we mean an absolutely continuous probability measure on the plane, by which this random point is distributed. For a triplet of random points, we can consider the possibly degenerate triangle, $\text{conv}\{x_0, x_1, x_2\}$, and ask questions about the probability of covering a fixed point p by this triangle.

In [17] (using the technique of M. Gromov [15]), it was proved that for a given triplet of random points in the plane, one can always find a point p such that the probability of the event $p \in \text{conv}\{x_0, x_1, x_2\}$ is at least $1/6$. However, the following question is still open:

Problem 4. Find the optimal constant in this theorem. Is it $1/6$ or a larger number?

In [12], it was shown that in the particular case when the random points have the same distribution, the optimal constant is $2/9$; in [12], discrete distributions were considered but this does not matter. In [17], the lower bound $2/9$ was also established for the case when two of the three distributions are the same. Intuitively, it seems that for different distributions, the constant should not be less than $2/9$, but no rigorous proof is known. In the bibliography, we give additional literature related to this problem.

4. Cutting a Convex Figure Into Six Pieces

There is a variety of results about ‘fair’ partitions of convex bodies or measures in Euclidean spaces. The famous Ham Sandwich Theorem [23, 22] asserts that any n absolute continuous probability measures in \mathbb{R}^n can be simultaneously partitioned into equal halves with a single hyperplane. In [21] (see also [16]), this result was generalized to the case when we want to partition n measures into m equal parts with a convex partition of \mathbb{R}^n .

Another almost elementary case of this problem is the partitioning a convex figure (compactum) in the plane into m parts of equal areas and perimeters. Nandakumar, Ramana Rao in [19] and Bárány, Blagojević, A. Szűcs in [10] considered particular cases, and [16] established the general result for prime powers $m = p^k$ using the technique of Victor Vasil’ev [24]. This result is relatively easy to establish for prime m , but beyond prime values of m , one cannot use a decomposition of m into prime factors (as was used in [21]) because the perimeter is *not* an additive function of convex bodies. So the simplest remaining case is:

Problem 5. Can one partition every convex figure $C \subset \mathbb{R}^2$ into six pieces of equal areas and perimeters?

5. Connected Algebraic Sets in the Plane

In [18], the proof of existence of spanning trees in the plane with low crossing with every line was simplified using the polynomial partitioning technique of Guth and Katz. This proof can be further simplified if one manages to answer the following question:

Problem 6. Prove that for any set of n points P in the plane there exists a polynomial f of degree at most $C\sqrt{n}$ such that P is contained in a single connected component of the set of zeros Z_f of f .

From dimensional considerations, it is easy to find a polynomial f of degree at most d satisfying $\binom{d+2}{2} > n$, whose zero set contains given n points. But the number of connected components of Z_f is bounded from above by $\binom{d-1}{2} + 1$ (this is the Harnack theorem [14]), so it is again approximately equals n . In order to give a positive solution to this problem, it would be sufficient to find a polynomial f of degree at most $C\sqrt{n}$ with $P \subset Z_f$ and the number of connected components of Z_f at most $(1 - \varepsilon)|P|$, for some small fixed $\varepsilon > 0$.

6. Minimizing the Sum of Squares

Problem suggested by O. Musin.

An olympiad problem (communicated by Fedor Petrov) asserts that given a finite point set P in the unit square, I^2 , it is possible to connect its points by a Hamiltonian cycle with the sum of squares of edge lengths at most 4. Evgeniy Shchepin noted that the cycles minimizing the sum of squares generate certain interesting Peano curves in the unit square.

A number of similar questions were discussed by Bern and Eppstein [11]. For example, it is true that for any set P of an even number of points in the d -dimensional cube I^d , it is possible to find a perfect matching (a partition into pairs) such that the sum of d -th powers of the lengths of matching segments is bounded by a constant C_d . Returning to the planar case and the sums of squares one may ask:

Problem 7. Let B be a disk of radius one in the plane. Is it true that any finite $P \subset B$ can be spanned by a Hamiltonian cycle with the sum of squares at most 8? Is it true that any finite P with even number of points has a perfect matching with sum of squares at most 4? What can be said if we substitute other convex figures for B and I^2 ?

It seems possible to do the case $|P| = 4$ by hand, the extremal configurations being inscribed quadrangles with perpendicular diagonals and inscribed triangles together with their orthocenters, as well as their limiting cases.

7. Fermat Points Construction and Melzak Algorithm

Problem stated in collaboration with A. Tuzhilin.

Fermat point, shortest trees, and 120° -property. The classical Fermat Problem asks for a point X in the Euclidean plane that minimizes the sum of distances from three fixed points A_i , which we write as $f(X) = |A_1X| + |A_2X| + |A_3X|$. The solution is referred as the *Fermat point*. The strict convexity of the distance function and, hence, the function f , implies the uniqueness of the Fermat point F for any triangle $\Delta = A_1A_2A_3$. If all the angles of the triangle Δ are less than 120° , then F coincides with the Torricelli point, and otherwise F is the vertex of the largest angle. Recall that the *Torricelli point*, T , is the intersection point of the circles circumscribing the equilateral triangles $A_iB_jA_k$ located

outside Δ and sharing its sides A_iA_j , or as the intersection point of three segments A_iB_i . In this case, T can be defined as the point such that all the angles A_iTA_j are equal to each other and, hence, equal to 120° . The resulting *shortest tree*, $G = \cup_i[A_iF]$, satisfies the 120° -*property*: the edges meet each other at angles greater than or equal to 120° .

Problem 8. Investigate the Fermat problem on the sphere, in the Lobachevski plane, on an Alexandrov surface of bounded curvature, in a normed plane. Is a Fermat point always unique? How can we construct the set of all Fermat points?

Remarks. The 120° -property remains valid in much more general situations: in Riemannian manifolds [25], in Alexandrov spaces of bounded curvature [26], and in particular, in the surfaces of polyhedra. Therefore, on the sphere, in the Lobachevski plane, and on an Alexandrov surface, the Fermat point can be determined similarly as a point from which the sides of the triangle are seen by an angle that is greater than or equal to 120° , but it is unclear how to construct it geometrically. In a normed plane, the uniqueness result is not valid in general.

Locally minimal networks and Melzak algorithm. More generally, there is the well-known Melzak algorithm that constructs *locally minimal trees*, which are plane trees consisting of straight segments meeting at angles greater than or equal to 120° ; see for example [27]. Given a binary tree G , a finite subset M of the plane, referred as boundary, and a bijection between M and the set of degree-1 vertices of G , then Melzak algorithm either constructs a locally minimal tree of type G joining M , or finds that such a tree does not exist. In the first stage of the algorithm, we reduce M and G as follows: (a) replace a pair of points m and m' from M such that the corresponding edges e and e' from G have a common vertex x by the third vertex w of the equilateral triangle $mm'w$, and (b) delete the edges e and e' from G and assign vertex x to the point w . As a result, we get a new binary tree with fewer boundary points. At the end of the first stage, we obtain a tree that consists of a single edge connecting two boundary points. The corresponding locally minimal tree is the straight segment. In the second stage of the algorithm, we reverse the direction and construct the tree step by step starting with the single straight segment. Each new degree-3 vertex, V , can be constructed as an intersection of the tree with the circle circumscribed around one of an equilateral triangle from the first stage.

Problem 9. Generalize the Melzak algorithm to the case of the Lobachevski plane, the sphere, an Alexandrov surface, a normed plane.

Remarks. In all the cases, it is not difficult to write a computer program that calculates the Fermat point or the locally minimal tree for a small number of boundary points numerically. We are interested in exact solutions, which are important for theoretical constructions. Here, even small numbers of points are challenging. For example, it is not known how to determine which of three possible locally minimal binary trees can be constructed on a four points subset in \mathbb{R}^3 .

8. Uniqueness of Steiner Minimal Tree

Problem stated in collaboration with A. Tuzhilin.

Steiner minimal trees. Let (X, ρ) be a metric space, V a finite subset of X , and $G = (V, E)$ a graph with vertex set V and edge set E . We say that G is a *graph* in (X, ρ) . If $e = \{x, y\} \in E$, then the value $\rho(e) = \rho(x, y)$ is said to be the *length* of the edge e . The sum of the lengths of all edges in E is the *length* of the graph G . Consider a finite subset $M \subset X$. If $G = (V, E)$ is a graph in (X, ρ) such that $M \subset V$, then we say that G *joins* or *connects* M . Write $\text{smt}(M)$ for $\inf \rho(G)$, taking the infimum over all trees G connecting M , calling it the *length* of a Steiner minimal tree for M . If $\rho(G) = \text{smt}(M)$, where G is a tree connecting M , then G is referred as a *Steiner minimal tree*, or a *shortest tree*, with *boundary* M .

The classical *Steiner Problem* asks for Steiner minimal trees for finite subsets of Euclidean plane. It has a long history, dating back to Fermat; see [27] and [32]. The local structure of shortest networks is described in many situations: on Riemannian manifolds [25], in normalized spaces [33], [34], [35], in Alexandrov spaces [26]. The global structure is not well understood. Generally, a boundary set permits several shortest networks — for example, the vertices of the square in the plane — but as it is proved in [36], a finite subset of the plane “in general position” (i.e. for an open everywhere dense set of n -element subsets of the plane) permits unique shortest tree.

Problem statement. In the case of Riemannian manifolds, the local structure of shortest trees is similar to the case of the plane. Namely, the edges of each such tree are shortest geodesic segments meeting in common vertices at angles greater than or equal to 120° . Also, all degree-1 vertices belong to the boundary, and without loss of generality, we can assume that all degree-2 vertices also belong to the boundary. It follows that the maximal degree of a vertex of a shortest tree is 3.

Problem 10. Prove the uniqueness of the shortest tree for a finite boundary set “in general position” in a Riemannian manifold.

Notice that the proof in [36] is based on the local structure of the shortest tree and on the geometry of the plane. Another proof suggested in [37] is more topological, deals with a wider class of locally minimal networks [32], but it cannot be extended to the general case. On the other hand, it seems likely that the problem can be answered in the affirmative.

Other spaces. The same question can be stated in other spaces, first of all in Alexandrov space of bounded curvature, and in particular on the surfaces of convex polyhedra. In general normed spaces, the result is not valid [33], and it is reasonable to investigate what properties of the normed space imply the uniqueness.

9. Existence of Shortest Networks in Banach Spaces

Problem stated by N. Strelkova.

Networks in Banach spaces. Here a *network* in a Banach space is just a finite set of segments. The natural graph structure on a network is defined in the following way. The *edges* are the segments, the *vertices* are the endpoints of the segments. If two segments share an endpoint, then the two corresponding edges share the corresponding vertex. A network is *connected* if the corresponding graph is connected. The *length*, $\ell(N)$, of a network N is the sum of the lengths of all its edges (segments), where the length of a segment $[a, b]$ is by definition $\|a - b\|$.

Let X be a Banach space and fix a finite set $A = \{a_1, a_2, \dots, a_n\} \subset X$. We say that a network N *connects* A if N is connected and all points from A are vertices of N . Note that N may also have vertices that are not in A ; we call them *additional* vertices.

Shortest networks. Consider the infimum of the length functional $\text{smt}(A) = \inf_N \ell(N)$ over the set of networks that connect A . The general question is whether this infimum is attained, i.e. whether there exists a *shortest network* — a network that connects A and is shorter than any other network connecting A . It is not difficult to prove that if X is reflexive then for any $A \subset X$ there exists a shortest network. The non-reflexive case is more complicated, and the shortest network may not exist (see below).

Shortest networks with one additional point. Consider

$$r_1(A) = \inf_{x \in X} \sum_{i=1}^n \|x - a_i\|.$$

The length is not minimized over the entire set of networks connecting A but over a subset, namely the networks with one additional point x and the set of edges $\{[a_i, x]\}_{i=1}^n$. Is the infimum $r_1(A)$ attained? If X is reflexive then the answer is positive. But in general this is not true, see [28], [29].

Connection between the two existence problems. If A consists of three points, then the problem of finding the shortest network is exactly the problem of finding an additional point x at which $r_1(A)$ is attained.

In [30], it was shown that there exists a Banach space X , and for any $n \geq 3$, a set A_n of n points in X , such that there is no shortest network connecting A . The idea of the proof is to take the three-point set A_3 from [29] such that $r_1(A_3)$ is not attained and prove that for every set A_n that is sufficiently close to A_3 in Hausdorff metric, there is no shortest network connecting A_n . (Recall that ε -close in Hausdorff metric means that for any $x \in A_3$ there exists a point $y \in A_n$ such that $\|x - y\| < \varepsilon$ and for any $y \in A_n$ there exists a point $x \in A_3$ such that $\|x - y\| < \varepsilon$.)

Problem 11. Let X be a Banach space. Suppose that for any three-point set $A_3 \subset X$, there exists a shortest network (i.e. $r_1(A_3)$ is attained). Is it true that for any $A \subset X$ there exists a shortest network?

10. Minimal Fillings of Finite Metric Spaces

Problems stated in collaboration with A. Tuzhilin.

Let M be a finite pseudo-metric space with distance function ρ , and let $G = (V, E)$ be a connected weighted graph with weight function ω such that $M \subset V$. Define another distance function d_ω that maps a pair of point u and v in M to the least possible weight of the path connecting the vertices u and v in G . We call G by a *filling* of the space M if for any two points u and v in M , the inequality $\rho(u, v) \leq d_\omega(u, v)$ is valid. By $\text{mf}(M)$, we denote the infimum of the weights $\omega(G)$ over all fillings G of the space M . A *minimal filling* satisfies $\omega(G) = \text{mf}(M)$. The theory of minimal fillings for finite metric spaces appears in [38] and [39].

Problem 12. Describe minimal fillings for the vertex sets of regular polygons in the Euclidean plane, on the standard sphere, and in the Lobachevskii plane.

Minimal fillings turns out to be closely related to shortest trees. It is proved in [39] that the weight of a minimal filling of a finite metric space M is equal to the infimum of the shortest trees spanning the images $\Phi(M)$ of the space M under isometric embeddings into all possible (compact) ambient metric spaces X . This infimum is attained, in particular, at the Kuratowskii embedding into (a ball in) the normed space $\mathbb{R}_\infty^{|M|}$ endowed with the max-norm $\|\cdot\|_\infty$, which is defined as $\|x\|_\infty = \max_i |x^i|$ for $x = (x^1, \dots, x^m)$. Recently Z. Ovsyannikov has shown that for an arbitrary finite boundary set, $M \subset \mathbb{R}_\infty^m$, the shortest tree connecting M in \mathbb{R}_∞^m gives its minimal filling.

Problem 13. Describe metric spaces X in which the shortest trees are minimal fillings for their boundaries.

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Open Problems in Discrete and Computational Geometry

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We have selected problems that may not yet be well known, but have the potential to push the research in interesting directions. In particular, we state problems that do not require specific knowledge outside the standard circle of ideas in discrete geometry. Despite the relatively simple statements, these problems are related to current research and their solutions are likely to require new ideas and approaches. We have chosen problems from different fields to make this short paper attractive to a wide range of specialists.

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