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# The Depth Poset of a Filtered Lefschetz Complex

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## Abstract

Taking a discrete approach to functions and dynamical systems, this paper integrates the combinatorial gradients in Forman's discrete Morse theory with persistent homology to forge a unified approach to function simplification. The two crucial ingredients in this effort are the *Lefschetz complex*, which focuses on the homology of a cell complex at the expense of the geometry of the cells, and the *shallow pairs*, which are birth-death pairs that can double as vectors in discrete Morse theory. The main new concept is the *depth poset* on the birth-death pairs, which captures all simplifications achieved through canceling shallow pairs. One of its linear extensions is the ordering by persistence.

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## 1 Introduction

The simplification of a smooth function on a manifold by canceling critical points in pairs is a classic idea in Morse theory [18], whose computational execution is riddled with technical difficulties. The 2- and 3-dimensional cases are of substantial practical importance in geometric visualization [12], and already in three dimensions, the technical challenges abound; see e.g. [11] or [16]. The purpose of this paper is to introduce new tools that help overcome the technical difficulties and facilitate a clean implementation of these topological ideas.

We begin with an intuitive introduction of the topological idea, which we illustrate with a real-valued function on a circle; see Figure 1, where A and I are identified. The graph of this function may be interpreted as a mountain range in the winter, with skiers populating its slopes. A skier who uses only the force of gravity can descend from a peak to one of the two adjacent valleys, but this is were the journey ends. The situation improves if there is a ski lift that leads up to a neighboring peak. Assuming the cost of constructing such a lift increases with the height difference, we build only one lift for each peak and valley, and only if it is the less expensive of the respective two choices for both, the peak and the valley; see the arrows in the upper left panel of Figure 1. From the skier's viewpoint, the lifts change the geometry of the mountain range as she can now reach further from most peaks. For example, from the peak labeled DE, she can now reach all the way to the valleys labeled C and G, but not yet beyond. The change in geometry can be visualized by leveling the peaks with lifts; see the upper right panel in Figure 1 for the outcome of this operation. The new geometry is reflected by the simplified graph, whose peaks and valleys are the ones without lifts. We iterate the construction of lifts and this way further extend the reach of our skier by simplifying the geometry; see the lower panels in Figure 1. The iteration ends with a single valley and a single peak that requires no additional lifts. In the original mountain range, every valley can now be reached from the remaining peak using the constructed lifts.


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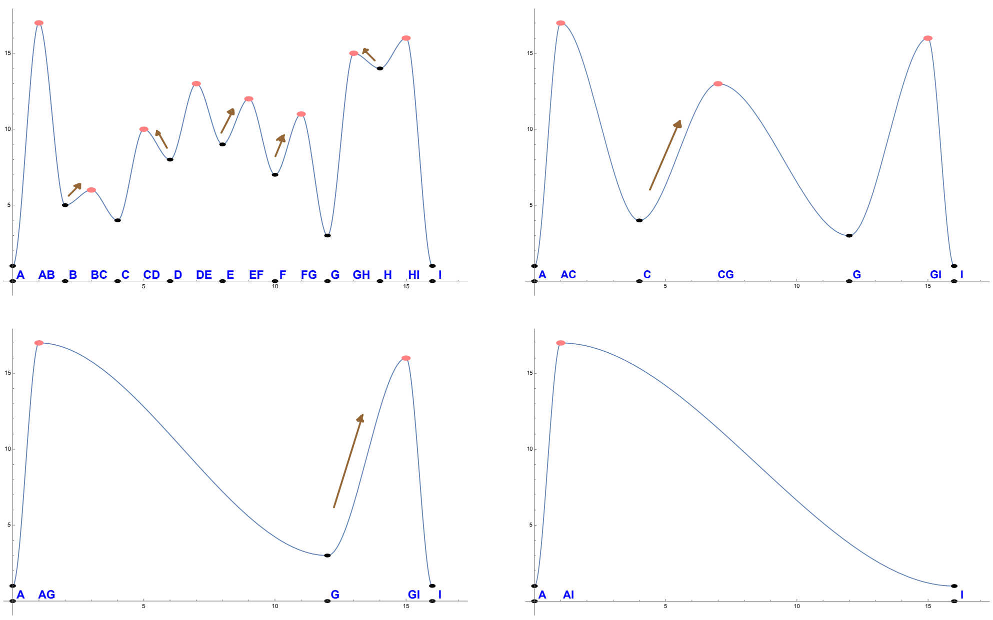


Figure 1: *Upper left:* a filter over the triangulation of a circle as height function. *Upper right, lower left, and lower right:* the three derived filters obtained by canceling the shallow pairs, which are indicated by upward sloping arrows.

In the hope to continue the development of a combinatorial theory of dynamics started in [19, 15], our approach is fundamentally discrete and not analytic. We therefore work with a cell complex that decomposes the space, which in the above example is a decomposition of the circle into edges that meet in pairs at shared vertices; see Figure 2. All we need to know

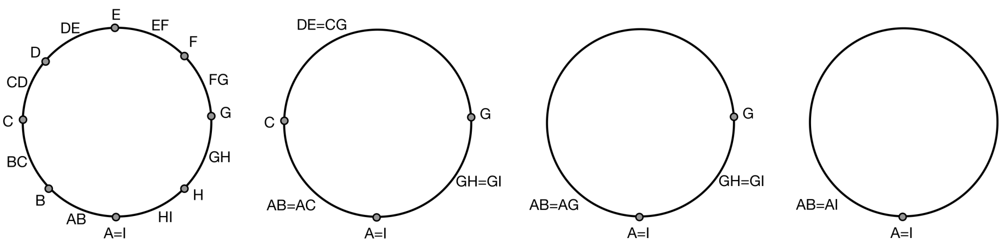


Figure 2: The triangulations of the circle whose vertices and edges correspond to the minima and maxima of the height functions in Figure 1. The cancellation of a peak-valley pair corresponds to removing the vertex of the valley and absorbing the edge of the peak into the neighboring edge.

about the geometry is the height of every peak, which we store with the corresponding edge, and the height of every valley, which we store with the corresponding vertex. The leveling of a peak corresponds to absorbing the corresponding edge into the neighboring edge that shares the vertex at which the ski lift originates. As a side-effect, this vertex disappears.

The illustrated 1-dimensional case is of course a gross over-simplication of the general situation, so it is important to mention that the idea and our methods generalize. The 2-dimensional case is illustrated in Figure 3, in which a real-valued function on the sphere on the left is represented by its critical points (minima, saddle points, maxima) and a few level lines to indicate the height differences. If we trace out the flow lines from the saddles down to the minima, we decompose the sphere into the *stable manifolds* of the function, which in the generic case of a Morse–Smale function is a cell complex; see the sphere on the right. The geometry of this cell complex can be challenging, in particular in higher

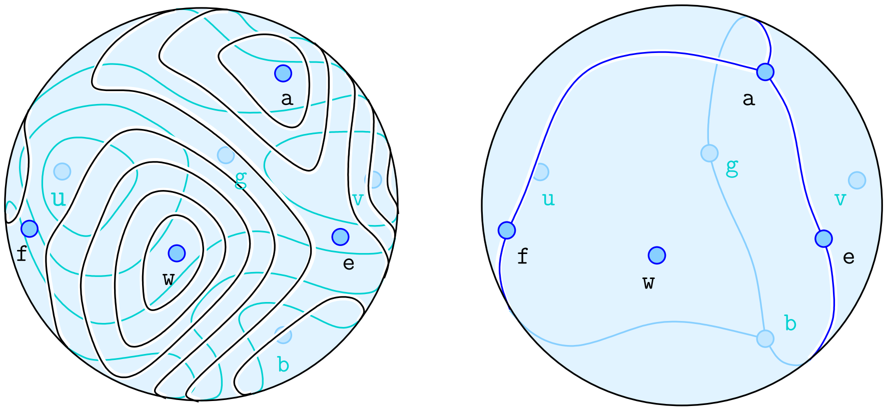


Figure 3: *Left:* level sets of a height function on the sphere. There are two minima, a,b, three saddles, e,f,g, and three maxima, u,v,w. *Right:* the flow lines from the minima to the saddle points, which trace the boundaries of the maxima's influence regions.

dimensions. We therefore separate the geometry from the topology, which is encapsulated in the incidence relation between the cells of different dimensions. This is formalized in the notion of a *Lefschetz complex*, which we introduce in Section 2. From a purely algebraic point of view, a Lefschetz complex is merely a basis of a free chain complex. However, we feel that it is important to advocate a different point of view, with the basis elements as first class objects, and the chain complex a tool built on top of them. In our setting, a Lefschetz complex may also be viewed as a data structure for storing the homology of the complex while ignoring the geometry of its cells. Section 3 describes how this homology changes when we cancel a critical point pair and how this affects the Lefschetz complex.

The remaining sections use these foundations to explore the simplification through successive cancellation. There is need for a global view in choosing the critical point pairs to cancel, else we may end in topologically convoluted dead-ends. We find guidance in the *vectors* of a combinatorial gradients [9], which we describe in Section 4, and in the *birth-death pairs* of persistent homology [7], which we explain in Section 5. The crucial concept that allows the unification of these two notions is that of a *shallow pair*, which is a sufficiently abundant type of birth-death pair that can be turned into a vector without sacrificing the acyclicity of the combinatorial vector field. By repeated cancellation of shallow pairs, we get a hierarchy of simplifications in terms of a partial order on the birth-death pairs, which we introduce in Section 6. Importantly, this poset does not depend on the order of the cancellations and represents all possible such simplifications as partitions of the poset into an upper set and a lower set.

We conclude this paper with a discussion of possible applications and open questions in Section 7. Among these is what originally motivated the authors of the paper to engage in this research, which is a multi-scale combinatorial theory of dynamics under local changes of the vector field. A first step would be a combinatorial Cerf theory, which restricts attention to acyclic vector fields; that is: to gradients.

## 2 Lefschetz Complexes

Introduced by the more modest name of a *complex* in the book on algebraic topology by Solomon Lefschetz [14], a Lefschetz complex may be understood as an abstraction of a cellular complex: its elements are the cells, and it stores the boundary relations between them. It is otherwise not concerned with the geometry of the cells, except that the homology of each cell is isomorphic to that of a pointed sphere of the same dimension; that is: the homology of the closed ball relative to its boundary; see the last paragraph of this section. We simplify its description by limiting ourselves to modulo-2 arithmetic.

▶ **Definition 2.1** (Lefschetz Complex). *A* Lefschetz complex *is a triplet* $(X, \dim, \kappa)$*, in which* $X$ *is a finite set of elements called* cells*,* $\dim \colon X \to \mathbb{N}_0$ *maps each cell to its dimension, and* $\kappa \colon X \times X \to \{0, 1\}$ *is a map such that* $\kappa(y, x) \neq 0$ *only if* $\dim y = \dim x + 1$*, and*

$$\sum\nolimits_{y \in X} \kappa(z, y) \cdot \kappa(y, x) = 0 \tag{1}$$

*holds for all* $z, x \in X$*. We call* $x$ *a* facet *of* $y$ *if* $\kappa(y, x) = 1$*.*

Referring to $y$ as a *p-cell* if $\dim y = p$, (1) says that for every $(p+1)$-cell, $z$, and every $(p-1)$-cell, $x$, there is an even number of $p$-cells, $y$, that are facets of $z$ and have $x$ as a facet. We will often say that $X$ is a Lefschetz complex, assuming dim and $\kappa$ are implicitly given.

As an example, consider the division of a 2-sphere into three wedges by connecting the two poles by three arcs; see Figure 4. There are two 0-cells, `a,b`, three 1-cells, `e,f,g`, and three 2-cells, `u,v,w`, and we have $\kappa = 1$ for the pairs `(w,e),(w,f),(v,g),(v,e),(u,f),(u,g),(g,a),(g,b),(f,a),(f,b),(e,a),(e,b)`. Note that (1) is satisfied throughout.

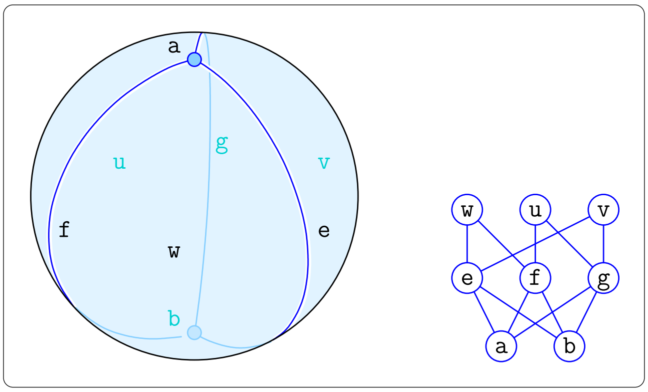


Figure 4: *Left:* a division of the 2-sphere into three wedges, in which `a,b` are the vertices at the north-pole and south-pole, `e,f,g` are the arcs that connect the poles, and `u,v,w` are the thus created regions. *Right:* the face poset of the division.

Since we use modulo-2 arithmetic, many of the common algebraic notions needed to define homology simplify to elementary combinatorial concepts. We write $C(X)$ for the set of subsets of $X$, and call $c \in C(X)$ a *chain*. It is *homogeneous* if all cells in $c$ have the same dimension, and we call it a *p-chain* if $c \neq \emptyset$ and the common dimension of the cells in $c$ is $p$. For $c, d \in C(X)$, we write $\langle c, d \rangle = \#(c \cap d)$ for the cardinality of their intersection.

The *boundary* of $y \in X$ is $\partial y \in C(X)$ consisting of all facets of $y$. Extending it linearly to chains, we get the *boundary homomorphism*, $\partial \colon C(X) \to C(X)$, which maps $c \subseteq X$

to $\partial c = \sum_{y \in c} \partial y$. Since we use modulo-2 arithmetic, the formal sum is the symmetric difference of the sets. Condition (1) guarantees $\partial\partial = 0$, so $(C(X), \partial)$ is a chain complex. $Z(X) \subseteq C(X)$ contains all *cycles*, which are the chains with zero (empty) boundary, and $B(X) \subseteq C(X)$ contains all *boundaries*, which are the chains that are the boundary of other chains. Since $\partial\partial = 0$, we have $B(X) \subseteq Z(X)$. The *homology* is the quotient of the two: $H(X) = Z(X)/B(X)$, which is the partition of $Z(X)$ into sets $B(X) + c$ with $c \in Z(X)$. This partition is well defined because $c, d \in Z(X)$ implies that $B(X) + c$ is either equal to or disjoint of $B(X) + d$. We say $X$ is *boundaryless* if $\kappa = 0$. Then $\partial = 0$ and $H(X)$ is the partition of $X$ into singletons.

The ordered pairs of cells in which the first is a facet of the second form a relation, and the transitive closure of this relation is a partial order, called the *face poset* of $X$. Indeed, we call $x$ a *face* of $z$, and write $x \leq z$, if there is a sequence of cells $x = y_0, y_1, \ldots, y_k = z$ such that $y_i$ is a facet of $y_{i+1}$ for $0 \leq i \leq k-1$. The case $k = 0$ is allowed, and we call $x$ a *proper face* of $z$ if $k \geq 1$. It induces a topology on $X$ via Alexandrov's theorem [1]: calling

- $U \subseteq X$ an *upper set* if $x \in U$ and $x \leq y$ implies $y \in U$;
- $L \subseteq X$ a *lower set* if $y \in L$ and $x \leq y$ implies $x \in L$,

the *open sets* in this topology are the upper sets, and the *closed sets* are the lower sets. We refer to this as the *Alexandrov topology* of $X$. This topology satisfies only the weakest of the separation axioms, namely that for any two distinct points, at least one has an open neighborhood that excludes the other. Indeed, if $x$ is a proper face of $y$, then every open set that contains $x$ also contains $y$, but not the other way round. Such a topological space is often referred to as a $T_0$ or Kolmogorov space. By a result of McCord [17], if the Lefschetz complex represents a regular cell complex (a cell complex with homeomorphic gluing maps), then its homology is isomorphic to the singular homology of that cell complex; see also Theorem 1.4.12 on simplicial complexes and Theorem 7.1.7 on regular complexes in [3]. An example in which the Lefschetz complex does not represent a regular cell complex is illustrated in the right panel of Figure 5. Since the Lefschetz complex remembers the dimensions of the cells, we get the homology of the 2-sphere, as desired. But note that this is different from the homology of the order complex of this poset, which consists of two points.

Let $(X, \dim, \kappa)$ be a Lefschetz complex, $Y \subseteq X$, and $\dim|_Y : Y \to \mathbb{N}_0$, $\kappa|_{Y \times Y} : Y \times Y \to \{0, 1\}$ the corresponding restrictions of dim and $\kappa$. Then we call $(Y, \dim|_Y, \kappa|_{Y \times Y})$ a *Lefschetz subcomplex* of $X$ if $Y$ is a Lefschetz complex. It is not difficult to see that if $Y$ is the intersection of an open set and a closed set in the Alexandrov topology of $X$, then $Y$ is a Lefschetz subcomplex. A particular example is a single cell, $x \in X$, which by itself is a Lefschetz complex. Having only one cell, this Lefschetz complex is boundaryless. It follows that its homology is zero, except in dimension $p = \dim x$, in which it is $\mathbb{Z}/2\mathbb{Z}$. This is the homology of a pointed sphere of dimension $p$, which thus proves the earlier claim about the cells in a Lefschetz complex.

## 3 Cancellations

We call an ordered pair of cells, $(s, t) \in X \times X$, a *reducible pair* in $X$ if $s$ is a facet of $t$. Given such a pair, we construct another Lefschetz complex in a process that may be viewed as a deformation retraction during which $a$ pulls the attached cells with it to attach to the other facets of $t$.

▶ **Definition 3.1** (Cancellation). *Given a reducible pair,* $(s,t)$ *in* $X$*, set* $X' = X \setminus \{s,t\}$*,* $\dim' = \dim|_{X'}$*, and* $\kappa' \colon X' \times X' \to \{0,1\}$ *defined by*

$$\kappa'(y,x) = \kappa(y,x) + \kappa(y,s) \cdot \kappa(t,x), \tag{2}$$

*call this operation the* cancellation *of* $(s,t)$*, and* $(X', \dim', \kappa')$ *the* quotient *of* $X$*.*

In words, we increment $\kappa(y,x)$ iff $s$ is a facet of $y$ and $x$ is a facet of $t$. See Figure 5 for examples of Lefschetz subcomplexes obtained through cancellations of reducible pairs of the Lefschetz complex in Figure 4.

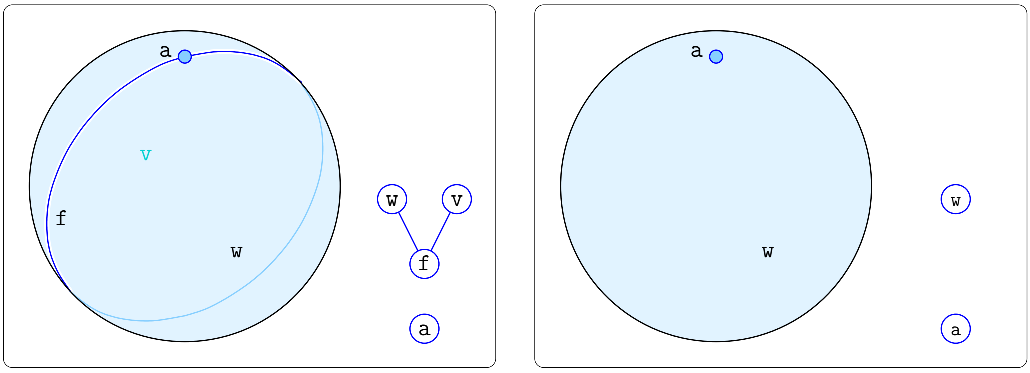


Figure 5: *Left:* the cancellations of the reducible pairs (`b`,`e`) and (`g`,`u`) simplifies the 3-division of the sphere in Figure 4 into a 2-division with a single vertex. *Right:* the further cancellation of the pair (`f`,`v`) leaves only two cells, the punctured sphere and the north-pole.

▶ **Lemma 3.2** (Quotient). *Let* $(s,t)$ *be a reducible pair in the Lefschetz complex* $(X, \dim, \kappa)$*. Then the quotient,* $(X', \dim', \kappa')$*, obtained by canceling* $(s,t)$ *is a Lefschetz complex, and the corresponding boundary homomorphism is defined by mapping a cell* $y \in X'$ *to*

$$\partial' y = \begin{cases} \partial y + \kappa(y,s)\partial t & \text{if } \dim' y = \dim t, \\ \partial y + \kappa(y,t) t & \text{if } \dim' y = \dim t + 1, \\ \partial y & \text{otherwise.} \end{cases} \tag{3}$$

The first row in (3) applies if $y$ and $t$ have the same dimension, and it removes the facets $y$ shares with $t$ and adds the other facets of $t$ as new facets of $y$. The second row applies if $y$'s dimension exceeds that of $t$ by one, and it removes $t$ as a facet of $y$, if it was such a facet. All other cells are unaffected. We note that $\kappa(y,s) = 0$ unless $\dim' y = \dim t$, and $\kappa(y,t) = 0$ unless $\dim' y = \dim t + 1$. Hence, (3) can be re-written as $\partial' y = \partial y + \kappa(y,s)\partial t + \kappa(y,t) t$.

**Proof.** We prove that the quotient is a Lefschetz complex by verifying that $\kappa'$ satisfies (1). While $\kappa'$ is formally defined only on $X'$, (2) makes sense also when $x = s$ and $y = t$, namely

$$\kappa'(y,s) = \kappa(y,s) + \kappa(y,s) \cdot \kappa(t,s) = 0, \tag{4}$$

$$\kappa'(t,x) = \kappa(t,x) + \kappa(t,s) \cdot \kappa(t,x) = 0 \tag{5}$$

because $\kappa(t,s) = 1$. We can therefore write the sum in (1) over all middle cells in $X$ rather than in $X'$:

$$\sum_{y\in X'}\kappa'(z,y)\kappa'(y,x) = \sum_{y\in X}[\kappa(z,y)+\kappa(z,s)\kappa(t,y)][\kappa(y,x)+\kappa(y,s)\kappa(t,x)] \tag{6}$$

$$= \sum_{y\in X}\kappa(z,y)\kappa(y,x)+\kappa(z,s)\sum_{y\in X}\kappa(t,y)\kappa(y,x)$$

$$+\kappa(t,x)\sum_{y\in X}\kappa(z,y)\kappa(y,s)+\kappa(z,s)\kappa(t,x)\sum_{y\in X}\kappa(t,y)\kappa(y,s), \tag{7}$$

which vanishes because each of the four sums in (7) vanishes by assumption of $X$ being a Leftschetz complex. We omit the proof of the boundary homomorphism, which is not difficult. ◀

We introduce three homomorphisms, which will be instrumental in proving properties of the quotient. The first expresses the cancellation of $(s,t)$ by mapping chains of $X$ to chains of $X'$, and the second goes the other direction, from $X'$ to $X$. The homomorphisms are $\pi\colon C(X)\to C(X')$, $\eta\colon C(X')\to C(X)$, and $\gamma\colon C(X)\to C(X)$ defined by

$$\pi(c) = c + \langle c,s\rangle\partial t + \langle c,t\rangle t; \tag{8}$$

$$\eta(c) = c + \langle \partial c,s\rangle t; \tag{9}$$

$$\gamma(c) = \langle c,s\rangle t. \tag{10}$$

We explain (8) in words: if $c$ contains $s$, then $\pi$ substitutes the other facets of $t$ for $s$, and if $c$ contains $t$, then $\pi$ deletes $t$. To explain (9), we note that $c$ is a chain in $X'$, so it is also a chain in $X$, and $\partial c$ denotes its boundary before the cancellation; that is: in $X$. If $\partial c$ contains $s$, then $\eta$ adds $t$ to the chain. Finally, if $c$ contains $s$, then $\gamma$ maps $c$ to $\{t\}$, and else it maps $c$ to the empty chain. Inspired by work in [13], we get the following:

▶ **Lemma 3.3** (Chain Homotopy). *The homomorphisms $\pi\colon C(X)\to C(X')$ and $\eta\colon C(X')\to C(X)$ defined by a reducible pair in the Lefschetz complex, $X$, are chain maps, and $\gamma\colon C(X)\to C(X)$ is a chain homotopy such that $\eta\circ\pi = \mathrm{id}_{C(X)}+\partial\circ\gamma+\gamma\circ\partial$ and $\pi\circ\eta=\mathrm{id}_{C(X')}$. In particular, the chain complexes $(C(X),\partial)$ and $(C(X'),\partial')$ are chain homotopic.*

**Proof.** To see the first relation, we apply first $\pi$ and then $\eta$ to $c\in C(X)$ and rewrite the terms using the two compositions of $\gamma$ and $\partial$:

$$\eta\circ\pi(c) = \eta(c+\langle c,s\rangle\partial t+\langle c,t\rangle t) \tag{11}$$

$$= c+\langle\partial c,s\rangle t+\langle c,s\rangle\partial t+\langle c,t\rangle t+\langle c,t\rangle t \tag{12}$$

$$= \mathrm{id}_{C(X)}(c)+\gamma\circ\partial(c)+\partial\circ\gamma(c), \tag{13}$$

since the last two terms in (12) cancel. To see the second relation, recall that $\pi\circ\eta$ applies to chains in $C(X')$, which by construction contain neither $s$ nor $t$:

$$\pi\circ\eta(c) = \pi(c+\langle\partial c,s\rangle t) \tag{14}$$

$$= c+\langle c,s\rangle\partial t+\langle c,t\rangle t+\langle\partial c,s\rangle t+\langle\partial c,s\rangle t \tag{15}$$

$$= \mathrm{id}_{C(X')}, \tag{16}$$

because the second and third terms in (15) vanish and the last two terms cancel. ◀

The existence of the chain homotopy asserted by Lemma 3.3 implies that the two Lefschetz complexes, $X$ and $X'$, have isomorphic homology.

## 4 Vectors of Combinatorial Gradients

Cancellations in a Lefschetz complex are not independent of each other, and one my enable or disable another. We use combinatorial gradients as introduced by Forman [9] to organize the cancellations and thus make their effect on the complex more predicbable. We begin by introducing the main notions and terminology, while referring to Forman [9, 10] for a more comprehensive treatment of the background.

Let $X$ be a Lefschetz complex. A *combinatorial vector field*, $V \subseteq X \times X$, is a collection of ordered pairs, called *vectors*, such that every cell belongs to at most one vector, and if $(s,t) \in V$, then $s$ is a facet of $t$. Every cell that does not belong to any vector is called a *critical cell*, while the vectors are made up of *non-critical cells*. There is an associated directed graph, $G_V$, whose vertices are the cells in $X$ and whose *explicit arcs* are the vectors in $V$. It also has *implicit arcs*, which are the pairs $(y,x)$ such that $x$ is a facet of $y$ but $(x,y)$ is not in $V$. A *(directed) path* is a sequence of vertices, $x_0, x_1, \ldots, x_n$ such that $(x_i, x_{i+1})$ is an arc in $G_V$ for $0 \leq i \leq n-1$. Its *length* is $n$, and the path is *trivial* if $n = 0$. A path is a *cycle* if $x_0 = x_n$ and $x_i \neq x_j$ for $0 \leq i < j < n$. Since the vectors are disjoint, every explicit arc of a path is followed by an implicit arc. Moving along an explicit arc, we gain one dimension, while moving along an implicit arc, we lose a dimension. A cycle ends at the same cell it started from, which implies that it alternates between explicit and implicit arcs.

We call $V$ a *combinatorial gradient* on $X$ if $G_V$ has only trivial cycles. A *Lyapunov function* for $V$ is a map $f\colon X \to \mathbb{R}$, such that $f(x) = f(y)$ whenever $(x,y)$ is an explicit arc, and $f(x) > f(y)$ whenever $(x,y)$ is an implicit arc. It follows that $f(x_0) \geq f(x_n)$ if there is a path from $x_0$ to $x_n$.

▶ **Lemma 4.1** (Lyapunov Function). *A combinatorial vector field on a Lefschetz complex admits a Lyapunov function iff it is a combinatorial gradient.*

**Proof.** "$\Longrightarrow$": if the vector field is not a gradient, then there is at least one non-trivial cycle from a cell $x_0$ back to $x_n = x_0$. After every explicit arc, there is an implicit arc, so this cycle contains at least one implicit arc. But this contradicts $f(x_0) = f(x_n)$.

"$\Longleftarrow$": since $G_V$ has no non-trivial cycle, the directed graph obtained by merging the endpoints of every explicit arc has an ordering such that $x$ precedes $y$ whenever $(x,y)$ is an implicit arc. Going from left to right in this ordering, we assign a strictly decreasing sequence of function values. If a vertex corresponds to the two endpoints of an explicit arc, both endpoints get the value of the vertex. This is a Lyapunov function because $f(x) > f(y)$ for every implicit arc $(x,y)$, and $f(x) = f(y)$ for every explicit arc $(x,y)$. ◀

An important property of a combinatorial gradient is the independence of the vectors if used in cancellations. We will see shortly, that this property crucially depends on the acyclicity of the associated diagraph.

▶ **Lemma 4.2** (Independence and Acyclicity). *Let $V$ be a combinatorial gradient on a Lefschetz complex, $X$, let $(s,t)$ be a vector in $V$, and write $X'$ for the quotient of $X$ through canceling $(s,t)$. Then $V' = V \setminus \{s,t\}$ is a combinatorial gradient on $X'$.*

**Proof.** We first prove that $V'$ is a combinatorial vector field on $X'$: if $(u,v) \neq (s,t)$ is a vector in $V$, then $u$ is still a facet of $v$ in $X'$. To see this, observe that at least one of $\kappa(v,s)$ and $\kappa(t,u)$ is zero, for else $s,t,u,v,s$ would be a non-trivial cycle in $G_V$. Hence, $\kappa'(v,u) = \kappa(v,u)$ by (2). Since $(u,v) \in V$, we have $\kappa(v,u) = 1$, so $\kappa'(v,u) = 1$, as claimed.

We second show that canceling $(s,t)$ preserves the acyclicity of the associated digraph. To derive a contradiction, assume that $G_{V'}$ has a non-trivial cycle and consider an arc $(y,x)$ in this cycle that is not arc in $G_V$. All explicit arcs of $G_{V'}$ are also explicit arcs of $G_V$, so $(y,x)$ is an implicit arc of $G_{V'}$ and $\kappa'(y,x) = 1$ while $\kappa(y,x) = 0$. Since $\kappa'(y,x) = \kappa(y,x) + \kappa(y,s)\kappa(t,x)$ by (2), this implies $\kappa(y,s) = \kappa(t,x) = 1$, so $y,s,t,x$ is a path in $G_V$. By replacing all such arcs $(y,x)$ in $G_{V'}$ by the paths $y,s,t,x$ in $G_V$, we obtain a non-trivial cycle in $G_V$, which contradicts $V$ being a combinatorial gradient. ◀

Lemma 4.2 implies that we can cancel all vectors in a combinatorial gradient, and this way obtain a Lefschetz complex in which only the critical cells remain. By Lemma 3.3, this Lefschetz complex is chain homotopic to $X$, and we will see shortly that it does not depend on the order in which the cancellations are applied. To this end, call a path in $G_V$ *regular* if the vertex at which two consecutive implicit arcs meet is necessarily critical. Write $\#(y,x)$ for the parity of regular paths from $y$ to $x$ in the associated digraph; that is: the number of such paths modulo 2.

▶ **Theorem 4.3** (Morse Complex). *Let $V$ be a combinatorial gradient on a Lefschetz complex, $(X, \dim, \kappa)$, and $(X'', \dim'', \kappa'')$ the Lefschetz complex obtained by canceling all vectors in $V$. Then $X''$ is the set of critical cells of $V$ in $X$, and for any two cells, $s,t \in X''$, we have*

$$\kappa''(t,s) = \#(t,s). \tag{17}$$

*So $(X'', \dim'', \kappa'')$ is independent of the order in which the vectors in $V$ are cancelled.*

**Proof.** The only part of the theorem that still needs proof is equation (17). Let $k = \dim t - \dim s$, which is the surplus of implicit arcs on any path from $t$ to $s$. Since $t$ and $s$ are critical, the first and last arcs are implicit, so the surplus is at least 1.

We first consider the case $k = 1$. The arcs in a path with surplus 1 alternate between implicit and explicit, which implies that every such path is regular. To prove (17), we use induction over the number of vectors in $V$, which we denote $n$. For $n = 0$, we have $X'' = X$ so every arc in $G_V$ is implicit. Hence, there is either no path from $t$ to $s$ or there is a path consisting of a single arc, in which case $s$ is a facet of $t$. Equivalently, $\kappa''(t,s) = \#(t,s)$, which establishes the induction basis.

For the induction step, let $V$ be a combinatorial gradient with $n \geq 1$ vectors and assume that (17) holds for all combinatorial gradients with $n-1$ vectors. Letting $(u,v)$ be a vector in $V$, we set $V' = V \setminus \{(u,v)\}$ and write $(X', \dim', \kappa')$ for the Lefschetz complex obtained by canceling the $n-1$ vectors in $V'$. By (2), we have

$$\kappa''(t,s) = \kappa'(t,s) + \kappa'(t,u) \cdot \kappa'(v,s) \tag{18}$$

for all $s,t \in X''$. Writing $\#'(t,s)$ for the parity of the regular paths from $t$ to $s$ in $G_{V'}$, we have $\kappa'(t,s) = \#'(t,s)$, $\kappa'(t,u) = \#'(t,u)$, and $\kappa'(v,s) = \#'(v,s)$ by induction. The only difference between the associated digraphs of $V$ and $V'$ is the arc connecting $u$ and $v$, which is explicit from $u$ to $v$ in $G_V$ and implicit from $v$ to $u$ in $G_{V'}$. Note that a path from $t$ to $s$ in $G_{V'}$ necessarily avoids this implicit arc. Indeed, if it used the arc from $v$ to $u$, then the preceding and succeeding arcs would also be implicit, but then the surplus of implicit arcs would be at least 2. It follows that $\#'(t,s)$ is the parity of the paths from $t$ to $s$ in $G_{V'}$ as well as of the paths from $t$ to $s$ in $G_V$ that avoid the explicit arc from $u$ to $v$. Furthermore, $\#'(t,u) \cdot \#'(v,s)$ is the parity of the paths from $t$ to $s$ in $G_V$ that use this explicit arc. Hence,

$$\#(t,s) = \#'(t,s) + \#'(t,u) \cdot \#'(v,s). \tag{19}$$

Comparing (18) with (19), we see that this implies $\kappa''(t,s) = \#(t,s)$, as desired. To finally prove (17) in the general case, we proceed by induction in $k$. The case $k = 1$ has already been established, so we fix $k \geq 2$ and assume (17) holds for all regular paths with surplus less than $k$. We have $0 = \kappa''(t,s) = \sum_x \kappa''(t,x)\kappa''(x,s)$ by definition. We restrict ourselves to vertices $x \in X''$ that satisfy $\dim x = \dim t - 1$, because otherwise $\kappa''(t,x)\kappa''(x,s) = 0$. By inductive assumption, we have $\kappa''(t,x) = \#(t,x)$ and $\kappa''(x,s) = \#(x,s)$. To complete the argument, we just need to ascertain that $\sum_x \#(t,x)\#(x,s) = \#(t,s)$. This is indeed the case because every regular path from $t$ to $s$ is the concatenation of a regular path from $t$ to $x$ and a regular path from $x$ to $s$, with $x$ being the first critical cell along the path different from $t$. This vertex, $x$, is different from $s$ because $k \geq 2$. ◀

Formula (17) in Theorem 4.3 shows that the quotient complex, $(X'', \dim'', \kappa'')$, is isomorphic to the Morse complex as constructed in [9, Section 8]. If $\kappa''(t,s) = 1$, then this implies a path from $t$ to $s$ in $G_V$ such that $t$ and $s$ are critical and all other cells along the path are non-critical. We refer to such paths as *connections*.

## 5 Shallow Pairs in Persistent Homology

We can simplify the Lefschetz complex beyond the Morse complex of the combinatorial gradient, but for this purpose, a different algebraic structure is needed as a guide. We use what we call the *depth poset* of the birth-death pairs. In a nutshell, it organizes the pairs such that any linear extension of the poset gives a valid sequence of cancellations. We begin with a brief introduction of persistent homology and refer to [7] for a more comprehensive treatment of the background.

By a *filter* of a Lefschetz complex, $X$, we mean an injection $\phi \colon X \to \mathbb{R}$ such that $\phi(x) < \phi(y)$ whenever $x$ is a proper face of $y$. Write $X_b = \phi^{-1}(\infty, b]$ for the sublevel set at $b \in \mathbb{R}$. By construction, every sublevel set of $\phi$ is a Lefschetz subcomplex of $X$. The increasing sequence of distinct sublevel sets is the *filtration* induced by $\phi$. To describe how the homology changes as we move from one sublevel set to the next, we write $[d]_b$ for the homology class of a cycle $d \in Z(X_b)$. Let $a < b$ be consecutive values of $\phi$; that is: there are cells $x, y \in X$ such that $a = \phi(x)$, $b = \phi(y)$, and $X_a = X_b \setminus \{y\}$. Since $\partial y$ is a boundary in $X_b$, $[\partial y]_b = 0$. We say $y$ *gives death* to a homology class if $[\partial y]_a \neq 0$. Otherwise, there is a chain $c \in C(X_a)$ such that $\partial c = \partial y$. It follows that $c + y$ is a cycle, and $[c+y]_b \neq 0$ because $c + y$ cannot be a boundary in $X_b$. In this case, we say $y$ *gives birth* to $[c+y]_b$. We write $X_*$ and $X_\times$ for the cells in $X$ that give birth and death of homology classes, respectively. Every cell does either, so $X_* \cap X_\times = \emptyset$ and $X_* \cup X_\times = X$. If $X$ is a boundaryless Lefschetz complex, then $X_* = X$ and $X_\times = \emptyset$.

Note that the homology class $[c+y]_b$ given birth to by $y$ is not uniquely determined. To fix this inconvenience, we observe that there is a unique chain $c_y \in C(X_a)$ such that $\partial c_y = \partial y$ and $c_y \subseteq X_\times$. Clearly, $y$ gives birth to the homology class of $d_y = c_y + y$. By construction, $d_y \cap X_* = \{y\}$, and we call $d_y$ the *canonical cycle* associated with $y$.

▶ **Lemma 5.1** (Canonical Cycle Basis). *Let $\phi \colon X \to \mathbb{R}$ be a filter on a Lefschetz complex, and $t \in \mathbb{R}$ a value of $\phi$. Then the canonical cycles $d_y$, with $y \in X_* \cap X_t$, form a basis of $Z(X_t)$.*

**Proof.** Let $d$ be a cycle in $X_t$, let $y_1, y_2, \ldots, y_k$ be the cells in $d$ that give birth in $\phi$, denote their canonical cycles by $d_1, d_2, \ldots, d_k$, and consider $d' = d_1 + d_2 + \ldots + d_k$. By construction, $d' \cap X_* = d \cap X_*$. To see $d' \cap X_\times = d \cap X_\times$, note that $d + d'$ is a cycle that contains no birth-giving cells. By construction, the death-giving cells do not form cycles, which leaves $d + d' = 0$ as the only possibility. Hence, $d = d'$, which implies that $d$ is a combination of the canonical cycles. ◀

To get a basis of $H(X_t)$, we need to identify the cells $y \in X_* \cap X_t$ whose canonical cycles have not been given death by any cell in $X_t$ yet. To do this, we construct a subset $Y_b \subseteq X_* \cap X_b$ such that the $[d_y]_b$, with $y \in Y_b$, form a basis of $H(X_b)$. The construction is inductive and paraphrases the original algorithm for computing persistent homology [8]. The induction proceeds along the linear ordering of the cells induced by the filter. Letting $b$ be the value of the first cell, $y$, we have $y \in X_*$ and set $Y_b = \{y\}$. For the inductive step, let $a < b$ be the values of two consecutive cells in the ordering, and let $y$ be the second cell, so $b = \phi(y)$. If $y \in X_*$, then $Y_b = Y_a \cup \{y\}$. Otherwise, $y \in X_\times$, which implies $[\partial y]_a \neq 0$. Since $Y_a$ defines a basis of $H(X_a)$, there is a unique subset $A \subseteq Y_a$ such that $d' = \sum_{x \in A} d_x$ satisfies $[d']_a = [\partial y]_a$. We let $z \in A$ be the cell with maximum value, write $\mathrm{birth}(y) = z$, and set $Y_b = Y_a \setminus \{\mathrm{birth}(y)\}$. We summarize for later reference.

▶ **Lemma 5.2** (Canonical Homology Basis). *Let $\phi \colon X \to \mathbb{R}$ be a filter on a Lefschetz complex, and $b \in \mathbb{R}$ a value of $\phi$. Then the $[d_y]_b$, with $y \in Y_b$, form a basis of $H(X_b)$.*

For every $y \in X_\times$, we call $(\mathrm{birth}(y), y)$ a *birth-death pair* of $\phi$, and we write $\mathrm{BD}(\phi)$ for the collection of birth-death pairs of the filter. It is easy to see that the thus constructed map, $\mathrm{birth} \colon X_\times \to X_*$, is injective. This implies that two birth-death pairs are either equal or they do not share any cell. Note however that birth is generally not bijective: cells in $X_*$ that are not in the image represent homology classes that never die, i.e. classes in $H(X)$.

As an example, consider the Lefschetz complex drawn in Figure 3 on the right, with cells $X = \{$`a,b,e,f,g,u,v,w`$\}$. Assuming the filter induces the alphabetic ordering of the cells, the birth- and death-giving cells are $X_* = \{$`a,b,f,g,w`$\}$ and $X_\times = \{$`e,u,v`$\}$, respectively. Correspondingly, we have three birth-death pairs: (`b,e`), (`g,u`), (`f,v`). The first two are reducible, while the third is not. This suggests we first cancel the two reducible pairs, hoping that these operations make the third pair reducible, and then cancel the third pair. This works in this particular example, but there are obstacles in the general case that require a finer distinction of the birth-death pairs, which we introduce next.

▶ **Definition 5.3** (Shallow Pairs). *Let $\phi \colon X \to \mathbb{R}$ be a filter on a Lefschetz complex. A pair of cells, $(s,t) \in X \times X$, is* shallow *if $s$ is a facet of $t$, $\phi(x) \leq \phi(s)$ for all facets $x$ of $t$, and $\phi(y) \geq \phi(t)$ for all cells $y$ that have $s$ as a facet.*

Equivalently, $(s,t)$ is a shallow pair if $s$ is the last facet of $t$ in the ordering induced by the filter, and $t$ is the first cell with facet $s$ in this ordering. We write $\mathrm{SH}(\phi)$ for the set of shallow pairs of the filter. Shallow pairs have been introduced in [5] under the name *apparent pairs*. They are more special than reducible birth-death pairs.

▶ **Lemma 5.4** (Shallow Pairs are Special). *Let $\phi \colon X \to \mathbb{R}$ be a filter on a Lefschetz complex. Every shallow pair of $\phi$ is a reducible birth-death pair, but not every reducible birth-death pair is necessarily a shallow pair of $\phi$.*

**Proof.** We first show that there are reducible birth-death pairs that are not shallow. Consider the Lefschetz complex in Figure 3, with a filter that induces the alphabetic order except that `v` precedes `u`. The birth-death pairs are (`b,e`), (`g,v`), (`f,u`), which are all reducible, but the third pair is not shallow because `f` is a facet of `u` as well as `v`, which precedes `u`.

We second prove that a shallow pair, $(s,t)$, is necessarily a reducible birth-death pair. Reducibility is immediate. To see that $(s,t)$ is a birth-death pair, set $a = \phi(s)$, $b = \phi(t)$, and recall that $[\partial t]_a \neq 0$. We have $s \in \partial t$, and since it is the last cell before $t$ in the linear ordering, $s \in Y_a$. Furthermore, $s$ belongs to the subset $A \subseteq Y_a$ for which $d' = \sum_{x \in A} d_x$ satisfies $[d']_a = [\partial t]_a$. Hence, $\mathrm{birth}(t) = s$, as claimed. ◀

A boundaryless Lefschetz complex has no shallow pair by definition. To see that every other filter has at least one shallow pair, let $t$ be the first cell in the linear ordering with $\partial t \neq 0$, and let $s \in \partial t$ be the last facet in this ordering. Clearly, $(s,t)$ is a shallow pair.

The remainder of this section justifies the introduction of shallow pairs by showing that their cancellation does not alter the persistent homology of the filtration other than in the obvious way. To state this in more technical terms, we use primes for all concepts that pertain to the quotient obtained by canceling a shallow pair.

▶ **Theorem 5.5** (Canceling a Shallow Pair). *Let $\phi\colon X \to \mathbb{R}$ be a filter on a Lefschetz complex, $(s,t)$ a shallow pair of $\phi$, and $\phi'\colon X' \to \mathbb{R}$ the filter on the quotient after canceling $(s,t)$. Then $\mathrm{SH}(\phi) \subseteq \mathrm{SH}(\phi') \cup \{(s,t)\}$ and $\mathrm{BD}(\phi) = \mathrm{BD}(\phi') \cup \{(s,t)\}$.*

The first claim is easily established, while the second is more demanding. In the interest of keeping with the flow of the current discussion, we move the proof of Theorem 5.5 to Appendix A, where the theorem is restated as two claims with separate arguments.

Theorem 5.5 exposes a weakness of the Lefschetz complex, which may alternatively be considered a strength, namely in overcoming limitations in simplifying functions on non-trivial spaces reported in [2]; see also [4]. Examples are the dunce hat—which has the homology of the disk but is not collapsible—and the Poincaré homology sphere—which is a 3-manifold that has the homology of the ordinary 3-sphere but is not homeomorphic to it. After canceling all birth-death pairs, the Lefschetz complex can no longer distinguish between the disk and the dunce hat, or between the ordinary 3-sphere and the Poincaré homology sphere, and this inability is crucial to cancel all birth-death pairs.

# 6 The Depth Poset

While Theorem 5.5 characterizes the impact of canceling a shallow pair on the persistent homology, we still need to understand the impact on the Lefschetz complex. To this end, we show that the collection of shallow pairs is a combinatorial gradient on the Lefschetz complex, so the quotient is well defined.

▶ **Lemma 6.1** (Shallow Pairs as Vectors). *Let $\phi\colon X \to \mathbb{R}$ be a filter on a Lefschetz complex. Then the set of shallow pairs, $\mathrm{SH}(\phi) \subseteq X \times X$, is a combinatorial gradient on $X$.*

**Proof.** We introduce $f\colon X \to \mathbb{R}$, which assigns the same values as the filter, except if the cell is in a shallow pair, in which case it assigns the smaller of the two values:

$$f(y) = \begin{cases} \phi(s) & \text{if } \exists (s,t) \in \mathrm{SH}(\phi) \text{ with } y = t, \\ \phi(y) & \text{otherwise.} \end{cases} \tag{20}$$

According to Lemma 4.1, if $f$ is a Lyapunov function, then $V = \mathrm{SH}(\phi)$ is a combinatorial gradient. To prove that $f$ is indeed such a function, we consider the associated digraph, $G_V$. Letting $x$ be a facet of $y$, the two cells are either connected by an explicit arc from $x$ to $y$, or by an implicit arc from $y$ to $x$. In the former case, $(x,y)$ is a shallow pair of $\mathrm{SH}(\phi)$, so we get $f(x) = f(y)$ by definition of $f$ in (20). In the latter case, we need to show that $f(x) < f(y)$. Since $f(x) \leq \phi(x)$ and $\phi(x) < \phi(y)$, by definition of filter, there is something to check only when $f(y) < \phi(y)$. In this case, there is a shallow pair, $(s,t)$, with $y = t$. Then $\phi(x) < \phi(s)$ because $x$ is a facet of $t$ but not the last one in the linear ordering induced by $\phi$, which is $s$. Since $f(x) = \phi(x)$ and $f(y) = \phi(s)$, this implies the required inequality. ◀

According to Lemma 6.1, we can cancel all shallow pairs, and then repeat for the quotient. By Theorem 5.5, the number of birth-death pairs decreases by the number of canceled shallow pairs. As mentioned earlier, the number of shallow pairs is strictly positive unless the Lefschetz complex is boundaryless. Hence, the number of birth-death points decreases from one iteration to the next, so the process ends after a finite number of iterations. Letting $k$ be this number, the iteration yields a sequence of Lefschetz complexes and filters on them:

$$\phi_j \colon X_j \to \mathbb{R}, \quad \text{for } 0 \leq j \leq k, \tag{21}$$

in which $\phi_0 \colon X_0 \to \mathbb{R}$ is $\phi \colon X \to \mathbb{R}$, and $\phi_j \colon X_j \to \mathbb{R}$ is the restriction of $\phi_{j-1}$ on the quotient of $X_{j-1}$ obtained after canceling all shallow pairs, for $j > 0$. We call $\phi_j$ the *$j$-th derived filter* of $\phi$. Correspondingly, we get a partition of the birth-death pairs of the initial filter into shallow pairs of the filters that arise during the iteration:

$$\mathrm{BD}(\phi) = \mathrm{SH}(\phi_0) \sqcup \mathrm{SH}(\phi_1) \sqcup \ldots \sqcup \mathrm{SH}(\phi_{k-1}). \tag{22}$$

This sequence is a hiearchy of simplifications of the original filter on a Lefschetz complex. A more refined hierarchy is obtained by identifying the subsets of shallow pairs that change the status of another birth-death pair from non-shallow to shallow. To define it, call a linear ordering of the birth-death pairs *cancelable* if each pair is shallow after canceling all its predecessors in the ordering. For example, every linear ordering in which all pairs in $\mathrm{SH}(\phi_{j-1})$ precede the pairs in $\mathrm{SH}(\phi_j)$, for $1 \leq j \leq k$, is cancelable. However, in general there are cancelable orderings that are not of this type. To cast list on them, we show that for each birth-death pair $(u,v)$ of $\phi_j$, there is a unique subset of shallow pairs of $\phi_{j-1}$, such that $(u,v)$ becomes shallow precisely at the moment all shallow pairs in this subset have been canceled. For any $S \subseteq \mathrm{SH}(\phi_{j-1})$, we write $\phi_{j-1}^S \colon X_j^S \to \mathbb{R}$ for the restriction of the filter to the quotient obtained by canceling the pairs in $S$.

▶ **Lemma 6.2** (Turning Shallow). *Let $\phi_{j-1}$ be the $(j-1)$-st derived filter of $\phi \colon X \to \mathbb{R}$, and $(u,v) \in \mathrm{SH}(\phi_j)$ a non-shallow birth-death pair of $\phi_{j-1}$. There is a unique $S_{(u,v)} \subseteq \mathrm{SH}(\phi_{j-1})$ such that for every $S \subseteq \mathrm{SH}(\phi_{j-1})$, $(u,v)$ is a shallow pair of $\phi_{j-1}^S$ iff $S_{(u,v)} \subseteq S$.*

**Proof.** By construction of the derived filters of $\phi$, $(u,v)$ is a shallow pair of $\phi_{j-1}^S$ if $S = \mathrm{SH}(\phi_{j-1})$; that is: when $\phi_{j-1}^S = \phi_j$. Write $V = \mathrm{SH}(\phi_{j-1})$, and consider what this means for the paths from $v$ to $u$ in the associated digraph, $G_V$. Since $\dim v = \dim u + 1$, each such path is an alternating sequence of implicit and explicit arcs, which are shallow pairs of $\phi_{j-1}$. Let $S_{(u,v)}$ be the subset of shallow pairs that belong to at least one path from $v$ to $u$.

Let $S \subseteq V$ and recall that canceling a pair $(s,t) \in S$ corresponds to replacing the explicit arc from $s$ to $t$ by the implicit arc from $t$ to $s$, and connecting any predecessor of $s$ directly to any successor of $t$. This shortens any path that contains $(s,t)$ by two arcs. If $S_{(u,v)} \subseteq S$, then canceling all pairs in $S$ shortens all paths from $v$ to $u$ to a single implicit arc. Equivalently, $(u,v)$ is a shallow pair of $\phi_{j-1}^S$. On the other hand, if $S_{(u,v)} \not\subseteq S$, then canceling the pairs in $S$ leaves at least one path of length at least 3 from $v$ to $u$. This path together with the arc from $u$ back to $v$ is a non-trivial cycle, which by Lemma 6.1 implies that $(u,v)$ is not a shallow pair of $\phi_{j-1}^S$. ◀

We return to the collection of cancelable linear orderings of the birth-death pairs of $\phi$. Each such ordering is a set of pairs, so taking the intersection is well defined.

▶ **Definition 6.3** (Depth Poset). *The* depth poset *of $\phi \colon X \to \mathbb{R}$, denoted* $\mathrm{Depth}(\phi)$*, is the intersection of all cancelable linear orderings of* $\mathrm{BD}(\phi)$*.*

By definition, Depth($\phi$) is the largest partial order such that every cancelable linear ordering is a linear extension of this poset. We claim that it is also the smallest partial order such that every one of its linear extensions is cancelable.

▶ **Theorem 6.4** (Cancelable Linear Orders). *Let $\phi\colon X \to \mathbb{R}$ be a filter on a Lefschetz complex. A linear ordering of* $\mathrm{BD}(\phi)$ *is cancelable iff it is a linear extension of* $\mathrm{Depth}(\phi)$.

**Proof.** By definition of the depth poset, every cancelable linear ordering of $\mathrm{BD}(\phi)$ is a linear extension of $\mathrm{Depth}(\phi)$. It thus suffices to the prove the converse.

To see that every linear extension of $\mathrm{Depth}(\phi)$ is cancelable, consider pairs $(s,t) \in \mathrm{SH}(\phi_{j-1})$ and $(u,v) \in \mathrm{SH}(\phi_j)$. By Lemma 6.2, they form a relation in the depth poset iff $(s,t) \in S_{(u,v)}$. Hence, $(s,t)$ precedes $(u,v)$ in every linear extension of $\mathrm{Depth}(\phi)$. This is true for any two birth-death pairs of consecutive derived filters. It is therefore easy to show inductively that every birth-death pair is shallow after all predecessors have been canceled. Equivalently, the linear extension is cancelable. ◀

To give an example, we return to the graph of the 1-dimensional function in Figure 1, which we interpret as a mountain range in winter. There are eight minima and eight maxima, and since all but the global minimum and the global maximum form pairs, we have seven birth-death pairs. Figure 6 shows the depth poset of these pairs, which in this case is a tree. Most of the cancellation sequences of shallow pairs produce partially simplified versions of

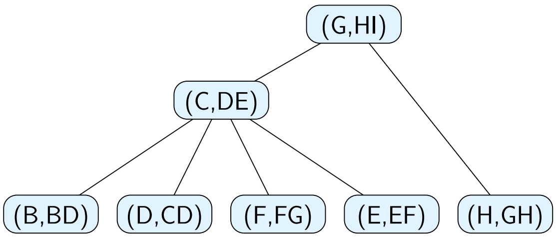


Figure 6: The transitive reduction of the poset on the birth-death pairs of the function whose graph is shown in the upper left panel of Figure 1.

the four derived filters. The exception is when (C, DE) precedes (H, GH). After canceling (C, DE) and before canceling (H, GH), the graph looks like the second derived filter to the left of G and the original filter to the right of G.

Define the *persistence* of a birth-death pair, $(u,v)$, as the absolute difference between their values: $\phi(v) - \phi(u)$. The persistence of a birth-death pair that is shallow during the first interation of constructing $\mathrm{Depth}(\phi)$ is not necessarily smaller than that of a birth-death pair that becomes shallow in later iterations. On the other hand, we will show that the pair with minimum persistence is shallow already in the first iteration. This implies that the ordering of the birth-death pairs by persistence is cancelable.

▶ **Theorem 6.5** (Ordering by Persistence). *Let $\phi\colon X \to \mathbb{R}$ be a filter on a Lefschetz. Then the ordering of the birth-death pairs by persistence is a linear extension of* $\mathrm{Depth}(\phi)$.

**Proof.** Assuming the minimum persistence birth-death pair is shallow, we can cancel it and iterate. This way we cancel the birth-death pairs in the order of their persistence, and since every cancelable ordering is a linear extension of the depth poset, $\mathrm{Depth}(\phi)$, so is the ordering by persistence.

To see that a minimum persistence birth-death pair is shallow we prove the contraposition; that is: a birth-death pair that is not shallow does not minimize persistence. Suppose

$(s,t) \in \mathrm{BD}(\phi)$ is not shallow, and fix a cancelable linear ordering in which $(s,t)$ comes immediately after the pair $(u,v) \in \mathrm{BD}(\phi)$, whose cancellation changes the status of $(s,t)$ from non-shallow to shallow. Let $X$ be the cells before canceling $(u,v)$, and set $V = \{(u,v)\}$, so the associated digraph, $G_V$ has a single explicit arc. The pair $(s,t)$ may or may not be reducible before canceling $(u,v)$. In the latter case, there is no arc connecting the two nodes, and because $(s,t)$ is reducible afterwards, we know from (17) that there is a path from $t$ to $s$ in $G_V$. Since there is only one explicit arc in $G_V$, the path must be $t,u,v,s$. Since $(u,v)$ is shallow at the time it is canceled, we have $\phi(s) < \phi(u) < \phi(v) < \phi(t)$, which shows that $(u,v)$ is a birth-death pair with smaller persistence than $(s,t)$.

There remains the case when $(s,t)$ is reducible before canceling $(u,v)$. Since $(s,t)$ is not yet shallow, there is a cell, $y$, with $\phi(y) < \phi(t)$ that has $s$ as a facet, or there is a facet, $w$, of $t$ with $\phi(s) < \phi(w)$. The two cases are symmetric. We therefore consider only the latter and assume that $w$ is the last facet of $t$ in the fixed linear ordering. We know that $(s,t)$ gets shallow eventually, so $w$ must be canceled prior to that event. Since $w$ is the last facet of $t$, it gives birth, so there exists a cell $z$ such that $(w,z)$ is a birth-death pair that becomes shallow before $(s,t)$. Hence, $\phi(s) < \phi(w) < \phi(z) < \phi(t)$, which implies that $(w,z)$ is a birth-death pair with strictly smaller persistence than $(s,t)$. ◀

## 7 Discussion

This paper introduces tools for the study of the dynamics under changing vector fields from a combinatorial viewpoint. Starting with the simpler case of a combinatorial gradient, it would be interesting to develop a combinatorial Cerf theory that classifies the non-generic critical cases, which necessarily arise when a vector field changes continuously. Perhaps the non-generic cases require multi-vectors consisting of more than two cells, which would go beyond the theory as introduced in [9]. This general topic is related to computing vineyards, as studied in [6]. At this time, the details of this relationship are unclear, primarily because of the different constraints imposed by the data structures representing the data, which are Lefschetz complexes in this paper and simplicial complexes in [6].

When we go to continuous vector fields more general than gradients, we observe recurring patterns that are more complicated than critical points, such as attractive or repulsive closed curves and more. The study of such phenomena may benefit from the ability to cancel a cell with one of its facets even if this creates a cell whose homology is different from that of a pointed sphere. Similarly, in the simplification of a smooth map, the restriction to cells with simple homology seems artificial and can sometimes be inconvenient.

We finally address the algorithmic aspects of the work described in this paper. From a computational point of view, the Lefschetz complex is an abstract data type that supports the cancellation of a reducible pair of cells. The repeated application of this operation shortens paths in the associated digraph, and the original paths between critical cells can be recovered by following the cancellations backward in time. All these operations reduce to the manipulation of lists and graphs, which are likely to have very fast implementations. It would be worthwhile to develop the details of these algorithms and to experiment with different data structures implementing them.

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# A Proof of Theorem 5.5

Theorem 5.5 makes two claims, one about the shallow pairs and the other about the birth-death pairs after the cancellation. We re-state and prove them separately in this appendix.

▷ Claim A.1 (Impact on Shallow Pairs). Let $\phi\colon X \to \mathbb{R}$ be a filter on a Lefschetz complex, $(s, t)$ a shallow pair of $\phi$, and $\phi'\colon X' \to \mathbb{R}$ the filter on the quotient after canceling $(s, t)$. Then $\mathrm{SH}(\phi) \subseteq \mathrm{SH}(\phi') \cup \{(s, t)\}$.

**Proof.** Let $(u, v) \in \mathrm{SH}(\phi)$ different from $(s, t)$. By Lemma 5.4 and the injectivity of the map birth: $X_\times \to X_*$, $u$ is different from $s$ and $t$ and so is $v$. Thus, $u$ is still a facet of $v$ in $X'$. It is possible that $v$ inherits additional facets from $t$, but only if $s$ is a facet of $v$, in which case all these new facets precede $s$ and therefore also $u$ in the linear ordering induced by $\phi$. Symmetrically, $u$ may inherit additional cells it is a facet of from $s$, but only if $u$ is a facet of $t$, in which case all these new cells succeed $t$ and therefore also $v$ in the linear ordering induced by $\phi$. Restricting this ordering to $X'$, we get the linear ordering induced by $\phi'$, which implies that $u$ is still the last facet of $v$, and $v$ is still the first cell $u$ is a facet of. Equivalently, $(u, v) \in \mathrm{SH}(\phi')$. ◀

▷ Claim A.2 (Impact on Birth-death Pairs). Let $\phi\colon X \to \mathbb{R}$ be a filter on a Lefschetz complex, $(s, t)$ a shallow pair of $\phi$, and $\phi'\colon X' \to \mathbb{R}$ the filter on the quotient after canceling $(s, t)$. Then $\mathrm{BD}(\phi) = \mathrm{BD}(\phi') \cup \{(s, t)\}$.

**Proof.** We write $y_1, y_2, \ldots, y_n$ for the linear ordering of the cells in $X$ induced by $\phi$. Let $k < \ell$ be the indices such that $s = y_k$ and $t = y_\ell$, and recall that removing $s$ and $t$ from this list gives the linear ordering of the cells in $X' = X \setminus \{s, t\}$ induced by $\phi'$. Let $c \in C(X)$ and recall that $\pi(c) \in C(X')$ is obtained by dropping $t$ and substituting the cells in $\partial t \setminus \{s\}$ for $s$; see the definition of this homomorphism in (8). The transition from $c$ to $\pi(c)$ is implicit in the operation that turns the Lefschetz complex of $X$ into the quotient, which is the Lefschetz complex of $X'$; see Definition 3.1. Since $(s, t)$ is a shallow pair, $s$ is the last facet of $t$ in the linear ordering or, equivalently, the other facets of $t$ precede $s$. It follows that the last cell of $c$ is also the last cell of $\pi(c) \cup \{s, t\}$ in the ordering. Hence,

$$c \in C(X_b) \Longrightarrow \pi(c) \in C(X'_b) \tag{23}$$

for any value $b \in \mathbb{R}$, in which we recall that $X_b \subseteq X$ and $X'_b \subseteq X'$ consist of all cells with value at most $b$. Following the original algorithm for computing persistent homology in [8], the remainder of this proof is inductive and establishes four hypotheses simultaneously. To formulate them, let $b = \phi(y_j)$ and write $d_j, d'_j$ for the canonical cycles associated with $y_j$, and $Y_j \subseteq X_* \cap X_b, Y'_j \subseteq X'_* \cap X'_b$ for the subsets of birth-giving cells such that the homology classes of their canonical cycles form bases of $H(X_b)$ and $H(X'_b)$, respectively; see Lemma 5.2. The hypotheses are

A. $d'_j = \pi(d_j)$ whenever $y_j \in X'_*$;
B. $y_j \in X_* \Rightarrow y_j \in X'_*$ and $y_j \in X_\times \Rightarrow y_j \in X'_\times$ whenever $y_j \in X'$;
C. $Y'_j = Y_j$ whenever $j < k$ or $\ell < j$, and $Y'_j = Y_j \setminus \{s\}$ whenever $k < j < \ell$;
D. $\mathrm{birth}'(y_j) = \mathrm{birth}(y_j)$ whenever $y_j \in X'_\times$.

By construction, $d_j \setminus \{y_j\} \subseteq X_\times$, which implies $s \notin d_j$ unless $s = y_j$. Hence, Hypothesis A is equivalent to $d'_j = d_j \setminus \{t\}$. Hypothesis B is equivalent to $X'_* = X_* \cap X'$ and $X'_\times = X_\times \cap X'$. By Lemma 5.4, we have $s \in Y_j$ iff $k \leq i < \ell$, which implies that Hypothesis C is equivalent to $Y'_j = Y_j \setminus \{s\}$ for all indices $j \neq k, \ell$. Hypothesis D readily implies that with the exception of $(s,t)$, the birth-death pairs are the same before and after the cancellation.

The four hypotheses are void and thus trivially true for $j = 0$, which serves as the induction basis. For the inductive step, consider a cell $y_j$, with value $b = \phi(y_j)$, and assume the four hypotheses are true for all indices $i \leq j-1$. Let $a = \phi(y_{j-1})$, and assume $y_j \neq s, t$, else there is nothing to prove.

Consider first the case $y_j \in X_*$, so the canonical cycle associated to $y_j$, denoted $d_j \subseteq (X_\times \cap X_a) \cup \{y_j\}$, is well defined. Let $d'_j = \pi(d_j)$. By construction, $y_j$ belongs to $d'_j$, and by (23) and the inductive assumption, all other cells in $d'_j$ belong to $X'_\times \cap X'_a$. Indeed, $s \notin d_j$ because it gives birth, so $d'_j \subseteq d_j$ because the cancellation of $(s,t)$ does not add any new cells to the cycle. Hence, $d'_j$ is the canonical cycle of $y_j$ in $X'$, which establishes Hypothesis A. But this also shows $y_j \in X_* \Rightarrow y_j \in X'_*$, which is Hypothesis B for birth-giving cells. In addition, it shows $Y_j = Y_{j-1} \cup \{y_j\}$ and $Y'_j = Y'_{j-1} \cup \{y_j\}$, which together with the inductive assumption implies Hypothesis C for birth-giving cells.

Consider second the case $y_j \in X_\times$. Then $\partial y_j$ is a non-trivial cycle in $X_a$. By (23), $\partial' y_j = \pi(\partial y_j)$ is a cycle in $X'_a$. To establish that it is non-trivial, assume there is a chain, $c_j \subseteq X'_a$ with $\partial' c_j = \partial' y_j$. We distinguish a few cases and conclude that $\partial y_j$ is trivial in each, which is a contradiction to the assumption and thus implies that $\partial' y_j$ is non-trivial.

1. $j < k$. Then $\partial' y_j = \partial y_j$ and $\partial' c_j = \partial c_j$. Hence, $\partial y_j$ is trivial, contradiction.
2. $k < j < \ell$. Then there cannot be any cycle homologous to $\partial y_j$ that contains $s$. Indeed, if there is such a cycle, then $s$ is a facet of a cell in $X_b$, which contradicts that $t$ is the first cell that has $s$ as a facet in the linear ordering. Hence, $\partial' y_j = \partial y_j$, and since $c_j \subseteq X_a$, we also have $\partial c_j = \partial y_j$, so again $\partial y_j$ is trivial, contradiction.
3. $\ell < j$. If $s, t \notin \partial y_j$, then we use the same argument as above to show that $\partial y_j$ is trivial, contradiction. This leaves two subcases.
   3.1 $s \in \partial y_j$. Then $\partial' y_j = \partial y_j + \partial t$. Hence, $\partial y_j = \partial' c_j + \partial t = \partial(c_j \cup \{t\})$, which implies that $\partial y_j$ is trivial, contradiction.
   3.2 $t \in \partial y_j$. Then $\partial' y_j = \partial y_j \setminus \{t\}$, and similarly, $\partial' c_j = \partial c_j \setminus \{t\}$. But then $\partial y_j = \partial c_j$, so $\partial y_j$ is trivial, contradiction.

In all cases, we get $y_j \in X'_\times$, which establishes Hypothesis B for death-giving cells. Next, we show that $y_j$ is paired with the same birth-giving cell before and after the cancellation; that is: $\mathrm{birth}'(y_j) = \mathrm{birth}(y_j)$. This will establish Hypothesis D, which then together with the inductive assumption establishes Hypothesis C. Let $A \subseteq Y_j$ be the birth-giving cells such that $d = \sum_{x \in A} d_x$ satisfies $[d]_a = [\partial y_j]_a$, and similarly let $A' \subseteq Y'_j$ be the birth-giving cells such that $d' = \sum_{x \in A'} d'_x$ satisfies $[d']_a = [\partial' y_j]_a$. In Case 1, we have $Y'_j = Y_j$ and $d'_i = d_i$ for every $y_i \in Y'_j$, so $A' = A$. In Case 2, we have $Y'_j = Y_j \setminus \{s\}$, but as argued there, $s \notin A$, which again implies $A' = A$. In Case 3, we have $Y'_j = Y_j$ and $d'_i = d_i \setminus \{A\}$ for every $y_i \in Y'_j$, so we get $A' = A$ in all subcases. We thus get $\mathrm{birth}'(y_j) = \mathrm{birth}(y_j)$ in all three cases.

This completes the inductive argument, which implies $\mathrm{BD}(\phi') = \mathrm{BD}(\phi) \setminus \{(s,t)\}$, as required to establish the claimed relation. ◀

# B Notation

| | |
|---|---|
| $(X, \dim, \kappa), (Y, \dim\|_Y, \kappa\|_{Y\times Y})$ | Lefschetz complex, subcomplex |
| $\partial\colon C(X) \to C(X)$ | boundary homomorphism |
| $C(X), Z(X), B(X), H(X)$ | chains, cycles, boundaries, homologies |
| $c, d$ | chains |
| | |
| $(X', \dim', \kappa')$ | quotient Lefschetz complex |
| $s, t; u, v$ | reducible pairs |
| $w, x, y, z$ | cells |
| $V, G_V$ | combinatorial gradient, associated digraph |
| $x_0, x_1, \ldots, x_n$ | path, connection, cycle |
| $f\colon X \to \mathbb{R}$ | Lyapunov function |
| | |
| $\phi\colon X \to \mathbb{R}$ | filter |
| $X_*, X_\times$ | birth-giving, death-giving cells |
| $X_b = \phi^{-1}(-\infty, b]$ | sublevel set |
| $Y_b' \subseteq Y_b$ | birth-giving cells alive at $b$ |
| $b = \phi(y); d_y; [d_y]_b$ | value; canonical cycle; homology class |
| | |
| $\mathrm{birth}\colon X_\times \to X_*$ | birth function |
| $\mathrm{BD}(\phi) = \{(\mathrm{birth}(y), y)\}$ | birth-death pairs |
| $\mathrm{SH}(\phi)$ | shallow pairs |
| $\mathrm{Depth}(\phi)$ | depth poset |
| $S, S_{(u,v)} \subseteq \mathrm{SH}(\phi_{j-1})$ | subsets of shallow pairs |
| $\phi_j\colon X_j \to \mathbb{R}, \phi_j^S\colon X_j^S \to \mathbb{R}$ | $j$-th derived filter, after canceling pairs |

Table 1: Notation used in the paper.

# C Results and Definitions

- Section 1: Introduction.
- Section 2: Lefschetz Complexes.
  - Definition 2.1 (Lefschetz Complex).
- Section 3: Cancelling Reducible Pairs.
  - Definition 3.1 (Cancellation).
  - Lemma 3.2 (Quotient).
  - Lemma 3.3 (Chain Homotopy).
- Section 4: Vectors of Combinatorial Gradients.
  - Lemma 4.1 (Lyapunov Function).
  - Lemma 4.2 (Independence and Acyclicity).
  - Theorem 4.3 (Morse Complex).
- Section 5: Shallow Pairs in Persistent Homology.
  - Lemma 5.1 (Canonical Cycle Basis).
  - Lemma 5.2 (Canonical Homology Basis).
  - Definition 5.3 (Shallow Pairs).
  - Lemma 5.4 (Shallow Pairs are Special).
  - Theorem 5.5 (Canceling a Shallow Pair).
- Section 6: The Depth Poset.
  - Lemma 6.1 (Shallow Pairs as Vectors).
  - Lemma 6.2 (Turning Shallow).
  - Definition 6.3 (Depth Poset).
  - Theorem 6.4 (Cancelable Linear Orders).
  - Theorem 6.5 (Ordering by Persistence).
- Section 7: Discussion.
- Appendix A: Proof of Theorem 5.5.
  - Claim A.1 (Impact on Shallow Pairs).
  - Claim A.2 (Impact on Birth-death Pairs).

## D To Think and to Do

- Section 1: Introduction.
- Section 2: Lefschetz Complexes.
- Section 3: Cancelling Reducible Pairs.
- Section 4: Vectors of Combinatorial Gradients.
- Section 5: Shallow Pairs in Persistent Homology.
- Section 6: The Depth Poset.
- Section 7: Discussion.
- Appendix A: Proof of Theorem 5.5.