The Poset of Cancellations in a Filtered Complex

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Abstract

Motivated by questions about simplification and topology optimization, we take a discrete approach toward the dependency of topology simplifying operations and the reachability of perfect Morse functions. Representing the function by a filter on a Lefschetz complex, and its (non-essential) topological features by the pairing of its cells via persistence, we simplify using combinatorially defined cancellations. The main new concept is the *depth poset* on these pairs, whose linear extensions are schedules of cancellations that trim the Lefschetz complex to its essential homology. One such linear extensions is the cancellation of the pairs in the order of their persistence. An algorithm that constructs the depth poset in two passes of standard matrix reduction is given and proven correct.

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1 Introduction

The primary aim of this paper is to shed light on the general question of simplification while preserving topology or, more specifically, on the dependencies between the operations that locally simplify. Examples are cancellations of critical point pairs in a Morse function, and collapses of simplex pairs in a simplicial complex. Depending on the sequence, these operations may or may not succeed in producing a perfect Morse function or a single vertex complex. Another source of motivation is the optimization of topology. To relate the two problems, we may think of 'simplifying' a function on a domain, while 'optimizing' the topology of a sublevel set of that function. The target of the optimization may address topology directly (such as minimizing the Betti numbers under some constraints) or indirectly (such as maximizing the strength-to-weight ratio of a shape). Optimizing shapes for everyday use is important, so there is a discipline within engineering dedicated to this subject [5].

The approach to these problems taken in this paper[1] is discrete and based on *Lefschetz complexes* [14] to represent shapes or spaces, which are abstractions of the more geometric cellular complexes. In this context, a continuous function is replaced by a filter, which maps cells to real numbers satisfying the mild requirement that the faces of a cell receive values

[1] A subset of the results appeared in an earlier version of this paper [10]. The connection to concepts in discrete Morse theory [12], and extensions to combinatorial dynamics [17] will be reported elsewhere.

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smaller than the cell. The operations are *cancellations* of cell in pairs, and preferably in *shallow pairs*—which were introduced under the name *apparent pairs* in [2]—as they preserve the rest of the topological structure to the extent this is possible.[2] The dependence between cancellations arises because pairs may or may not become shallow depending on which shallow pairs are canceled in which sequence. These dependencies are captured by the *depth poset*, which we construct using customized matrix reduction algorithms, and which may be used to annotate the persistence diagram of the filter. Figure 1 shows an example in the simplistic setting of a function on a circle: three rounds of cancellations of shallow min-max pairs suffice to produce a function with a single min-max pair, and the poset at the lower right presents all linear schedules of shallow cancellations.

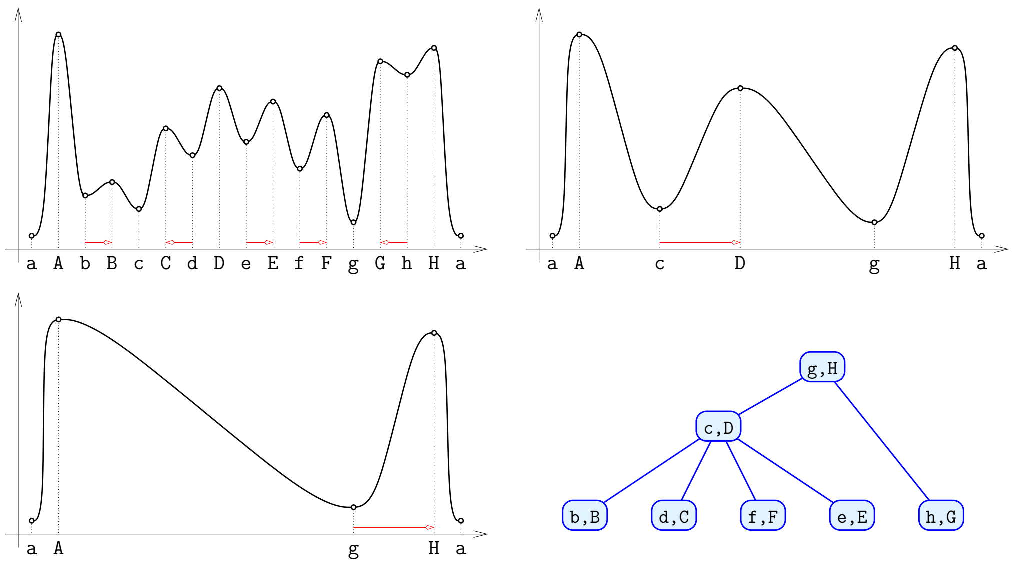

Figure 1: *Upper left:* a generic smooth function with 8 minima and 8 maxima on a circle. *Upper right:* simplified version of the function after canceling all 5 shallow min-max pairs, which are indicated by *red* arrows. The 5 cancellations turn a former non-shallow min-max pair shallow, whose cancellation leads to the further simplified version of the function at the *lower left*. The cancellation of the last birth-death pair, which is now shallow, produces a function with a single minimum and a single maximum (not shown). *Lower right:* the depth poset, whose relations express the dependencies between the cancellations: its linear extensions are sequences such that each pair is shallow at the time it is canceled.

There is related prior work on simplifying piecewise linear functions using the persistence diagram to quantify distortion, which gives satisfying results for 2-manifolds but runs into topological obstacles for 3-manifolds [1, 4]. The prior work on topological optimization most directly related to this paper has focused on operations that move points in the persistence diagram, which include cancellations [11, 18]. The customized matrix reduction we use to construct the depth poset uses elements of the column and row reduction algorithms for persistent homology described in [3, 7].

Outline. Section 2 explains Lefschetz complexes and cancellations. Section 3 introduces shallow pairs as special birth-death pairs defined in persistent homology. Importantly, it identifies two special total orders along which all cancellations are of shallow pairs. Section 4

[2] In the case of a 1-dimensional function, a min-max pair is shallow iff the max is the lower of the two neighboring maxima of the min, and the min is the higher of the two neighboring minima of the max. The structure of these pairs was recently exploited in adaptive sorting of lists [19].

defines the main new concept, the depth poset, proves some of its properties, and gives a matrix reduction algorithm to construct it. One of the off-shots of this construction is the insight that the order of the birth-death pairs by persistence also enjoys the property mentioned for the two special total orders. Finally, Section 5 concludes the paper.

2 Cancellations in Lefschetz Complexes

We are interested in the dependence of the topological features of a function on a space or, in the discrete setting studied in this paper, of a filter on a complex. To make this concrete, we need to specify what we mean by a feature, and what family of complexes and operations between them we consider. This section fixes the latter two variables to the Lefschetz complexes and cancellations between them, while it leaves the discussion of the features to the next section.

2.1 Lefschetz Complexes

We work with an abstraction of a geometric cellular complex, referred to as a Lefschetz complex. It keeps track of the dimension of each cell and its incidences with cells of one lower or higher dimension, but it does not worry about geometric details, such as how the cells are attached to each other. To simplify its exposition, we use modulo-2 arithmetic throughout this paper, which amounts to working with homology for coefficients in $\mathbb{Z}/2\mathbb{Z}$.

▶ **Definition 2.1** (Lefschetz Complex). *A* Lefschetz complex *is a triplet* $(X, \dim, \Delta)$*, in which* X *is a finite set of elements called* cells*,* $\dim\colon X \to \mathbb{Z}$ *maps each cell to its dimension, and* $\Delta\colon X \times X \to \{0,1\}$ *is a map such that* $\Delta(x,y) \neq 0$ *only if* $\dim y = \dim x + 1$*, and*

$$\sum\nolimits_{y \in X} \Delta(x,y) \cdot \Delta(y,z) = 0 \tag{1}$$

holds for all $x, z \in X$*. If* $\Delta(x,y) = 1$*, we call* x *a* facet *of* y*, we call* y *a* cofacet *of* x*, and we write* $x < y$ *to denote this relation. The* dimension *of* X *is* $\dim X = \max_{x \in X} \dim x$*.*

We will sometimes shorten the notation and refer to X as a Lefschetz complex. Using Equation (1), we associate with X a chain complex and homology, following the same standard scheme as for cellular complexes. Reusing the notation Δ for the associated *boundary matrix*, we observe that $\Delta[x,y] = \Delta(x,y)$. Whenever convenient, we split Δ into the boundary matrices dedicated to individual dimensions, with Δ_p recording the incidences between cells of dimension p and $p-1$.

The abstraction of a cellular complex to its Lefschetz complex is with controlled loss of information. Beyond the geometric details, we also lose information about the homotopy type. An example is the 3-sphere, which may be represented by the Lefschetz complex consisting of two isolated cells, one of dimension 0 and the other of dimension 3. The same Lefschetz complex represents the Poincaré homology 3-sphere, which has isomorphic homology groups but a different homotopy type than the 3-sphere [13].

A simplicial complex and its barycentric subdivision have identical underlying spaces and therefore isomorphic homology groups. Abstractly, the barycentric subdivision corresponds to the *order complex* of the face poset of the simplicial complex. This observation generalizes to *regular complexes*, whose cells are topological balls attached to each other via homeomorphic gluing maps, but not necessarily to cellular complexes with more complicated gluing maps. Alternatively, we can consider the free chain complex defined by the Lefschetz complex and define its homology from the corresponding cycle and boundary groups. While the thus obtained homology of a Lefschetz complex is generally different from the singular homology of the corresponding order complex, the two agree when the Lefschetz complex represents a regular complex. Indeed the following is a corollary of a theorem by McCord from 1966.

▶ **Theorem 2.2** (McCord [16]). *If X is a regular complex, then the homology of the free chain complex defined by its Lefschetz complex is isomorphic to the singular homology of X.*

The following two subsections give the reasons we will work with the homology of the free chain complex, also for cases in which the Lefschetz complex does not correspond to a regular complex, such as the ones in Figure 3.

2.2 Cancellations

Intuitively, a cancellation in a complex is like a collapse, except that it can also happen inside and thus away from the boundary. Such an "interior collapse" has consequences, as it distorts cells and may turn a regular complex into one in which the gluing maps are no longer homeomorphic. We cope with these consequences by ignoring them on the account of the more abstract Lefschetz complex.

▶ **Definition 2.3** (Cancellation). *Let $(X, \dim, \Delta)$ be a Lefschetz complex and $s < t$ both in X. The* cancellation *of the pair removes both cells and updates the incidence relation accordingly. Specifically, it sets $X' = X \setminus \{s, t\}$, $\dim' = \dim|_{X'}$, and $\Delta' \colon X' \times X' \to \{0, 1\}$ such that*

$$\Delta'(x, y) = \Delta(x, y) + \Delta(s, y) \cdot \Delta(x, t), \quad (2)$$

for all $x, y \in X'$. We refer to $(X', \dim', \Delta')$ as the quotient *after canceling s and t.*

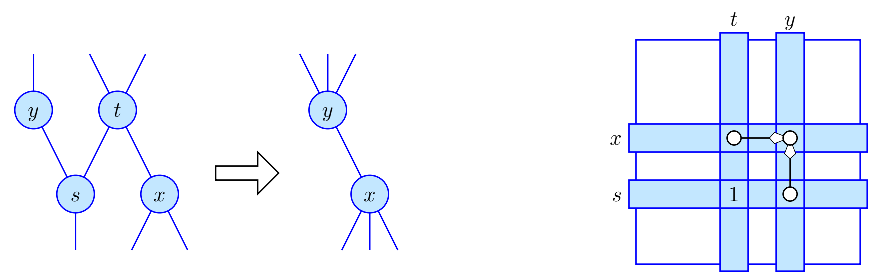

Figure 2: The effect of canceling $s < t$ on the Lefschetz complex on the *left* and the boundary matrix on the *right*. If in addition x were also incident to y, then the cancellation would removed this incidence, leaving y without child and x without parent (not shown).

Figure 2 illustrates the effect of canceling $s < t$. In particular, the cancellation adds column t to every other column y for which $s < y$ or, alternatively, it adds row s to every other row x for which $x < t$. After either the column or the row operations, the cancellation removes rows s and t as well as columns s and t from the matrix. It is not difficult to see that the quotient is again a Lefschetz complex. More importantly, the cancellation preserves the homology of the complex, since it translates into row or column operations that preserve the ranks of the individual boundary matrices. We state this for later reference.

▶ **Proposition 2.4.** *A Lefschetz complex and its quotient after canceling a facet-cofacet pair have isomorphic homology groups.*

Example. To show that cancellations in a Lefschetz complex are more powerful than collapses, we illustrate how they remove a Dunce hat [21] attached to one end of a cylinder in Figure 3.

To get the ranks of the homology groups, we count the generators of the cycle and boundary groups; see Table 1. As predicted by Proposition 2.4, the cancellation does not affect the ranks of the homology groups.

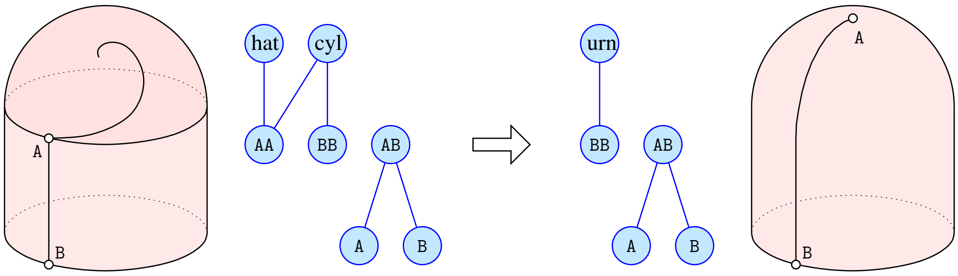

Figure 3: *Far left:* a cylinder cut along the edge `AB`, which connects the points `A` and `B` on its two boundary circles (represented by the edges `AA` and `BB`), and (an artistic sketch of) a Dunce hat attached to `AA` three times. *Far right:* after canceling the Dunce hat and `AA`, we get an upside-down urn cut along the edge connecting `A` (to which the Dunce hat contracted) to `B`. *In the middle:* the Lefschetz complexes before and after the cancellation of the Dunce hat.

	before			after		
	Z_p	B_p	β_p	Z_p	B_p	β_p
$p = 0$	`A,B`	`A+B`	1	`A,B`	`A+B`	1
$p = 1$	`AA,BB`	`AA,BB`	0	`BB`	`BB`	0
$p = 2$	$\emptyset$	$\emptyset$	0	$\emptyset$	$\emptyset$	0

Table 1: The generators of the cycle and boundary groups of the Lefschetz complexes in Figure 3. Recall that these complexes differ by canceling the Dunce hat at the top of the space on the *left*.

3 Shallow and Other Birth-death Pairs

In this section, we return to the notion of a topological feature, which for a filtered complex will be a birth-death pair of cells. We define them in a quick introduction to persistent homology and refer to [8] for more comprehensive background on this topic. Among the birth-death pairs, we will single out the simplest kind as shallow pairs, which we use to explore the dependence between all birth-death pairs of a given ordered complex.

3.1 Persistent Homology

By a *filter* of a Lefschetz complex, X, we mean an injective function $f\colon X \to \mathbb{R}$ such that $f(x) < f(y)$ whenever $x < y$. Write $X_b = f^{-1}(-\infty, b]$ for the sublevel set at $b \in \mathbb{R}$. By construction, every sublevel set of f is a Lefschetz complex, and we refer to the increasing sequence of distinct sublevel sets as the *filtration* induced by f. To describe how the homology changes as we move from one sublevel set to the next, we write $[d]_b$ for the homology class of a cycle $d \in \mathsf{Z}(X_b)$. Let $a < b$ be consecutive values of f, so there are cells $x, y \in X$ such that $a = f(x)$, $b = f(y)$, and $X_b = X_a \cup \{y\}$. Since ∂y is a boundary in X_b, $[\partial y]_b = 0$, and if $[\partial y]_a \neq 0$, then we say y *gives death* to a homology class. Otherwise, there is a chain $c \in \mathsf{C}(X_a)$ such that $\partial c = \partial y$. In this case, $c + y$ is a cycle, and $[c + y]_b \neq 0$ because X_b contains no cofacet of y yet, so c_y cannot be a boundary in X_b. We therefore say y *gives birth* to $[c + y]_b$. We write X° and $X^\times$ for the cells in X that give birth and death respectively. Every cell does either, so $X^\circ \cap X^\times = \emptyset$ and $X^\circ \cup X^\times = X$.

We need additional notions to associate births with deaths. First note that the homology class $[c+y]_b$ given birth to by y is generally not unique. To fix this inconvenience, we observe that there is a unique chain $c_y \in \mathsf{C}(X_a)$ such that $\partial c_y = \partial y$ and $c_y \subseteq X^\times$. Clearly, y gives birth to the homology class of $d_y = c_y + y$, and we call d_y the *canonical cycle* associated with y. Following the original persistence algorithm in [9], we use an inductive argument to define the pairing and simultaneously a set $Y_a \subseteq X_a$, which we will see contains all birth-giving cells that are not yet paired. Initially, both these sets are empty. For the inductive step, assume we have $Y_a \subseteq X_a$, and let b be the next value, after a, and y the next cell, with $f(y) = b$. If $y \in X^\circ$, we set $Y_b = Y_a \cup \{y\}$. Otherwise $y \in X^\times$, which implies $[d_y]_b \neq 0$. There is a unique subset $A \subseteq Y_a$ such that $d' = \sum_{x \in A} d_x$ satisfies $[d']_a = [\partial y]_a$. We let z be the last cell in A (the cell with maximum value), write $\mathrm{bth}(y) = z$, and set $Y_b = Y_a \setminus \{\mathrm{bth}(y)\}$. This defines an injective map, $\mathrm{bth}\colon X^\times \to X^\circ$, but note that it is not necessarily bijective since there may be cells in X° that never die. They represent the homology of X.

▶ **Definition 3.1** (Birth-death Pairs). *Let $f\colon X \to \mathbb{R}$ be a filter on a Lefschetz complex. Then $(s,t) \in X \times X$ is a* birth-death pair *of f if $s = \mathrm{bth}(t)$.*

Example. Consider the function on the circle displayed in the upper left panel in Figure 1. Representing the function by a filter, we let each minimum be a vertex, whose filter value is the function value (height) of the minimum, and each maximum an edge, whose filter value is the height of the maximum. The birth-death pairs marked by arrows in the upper left panel are $(\mathtt{b},\mathtt{B}), (\mathtt{d},\mathtt{C}), (\mathtt{e},\mathtt{E}), (\mathtt{f},\mathtt{F}), (\mathtt{h},\mathtt{G})$, and the remaining birth-death pairs marked by arrows in the upper right and the lower left panels are $(\mathtt{c},\mathtt{D})$ and $(\mathtt{g},\mathtt{H})$. As we will see shortly, the first five birth-death pairs are shallow, and the last two are not. The remaining two critical points, the minimum `a` and the maximum `A`, both give birth and are unpaired as they represent the homology of the circle (one component and one 1-cycle).

3.2 Shallow Pairs

A birth-death pair, $(s,t) \in \mathrm{BD}(f)$, can be cancelled if s is a facet of t in the Lefschetz complex. To avoid that this cancellation affects other birth-death pairs, we limit ourselves to canceling only special such pairs.

▶ **Definition 3.2** (Shallow Pairs). *Let $f\colon X \to \mathbb{R}$ be a filter on a Lefschetz complex. A pair $(s,t) \in X \times X$ is* shallow *if s is the last facet of t and t is the first cofacet of s in the filter, and we write $\mathrm{SH}(f)$ for the set of shallow pairs of the filter.*

In other words, $(s,t) \in \mathrm{SH}(f)$ if $f(x) \leq f(s)$ for all $x < t$ and $f(y) \geq f(t)$ for all $y > s$. We use the *ordered boundary matrix*—whose rows and columns are sorted by filter values—to recognize when $s < t$ is a shallow pair, or just a birth-death pair. With reference to the left panel of Figure 4, we write $r(s,t)$ for the rank of the lower left minor obtained by deleting all rows above row s and columns to the right of column t in the boundary matrix, and we let u be the row right after (below) row s and v the column right before (to the left of) column t. The following proposition is the Pairing Uniqueness Lemma in [6] restated for Lefschetz complexes:

▶ **Lemma 3.3** (Cohen-Steiner et al. 2006). *Let $f\colon X \to \mathbb{R}$ be a filter on a Lefschetz complex, and $s,t \in X$ two of its cells. Then $(s,t) \in \mathrm{BD}(f)$ iff $r(s,t) - r(s,v) - r(u,t) + r(u,v) > 0$.*

Compare this with a shallow pair in which row s is zero to the left of column t, and column t is zero below row s; see the right panel of Figure 4. Assuming $s < t$ is shallow, the ranks of

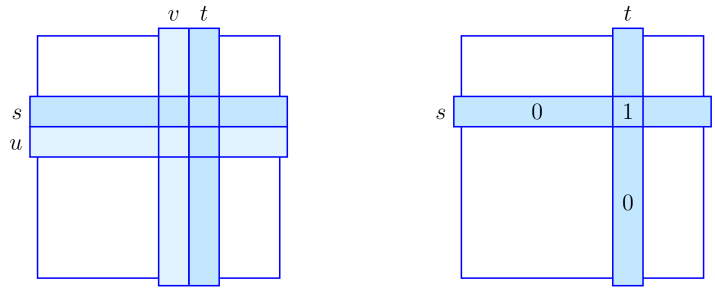

Figure 4: *Left:* the $s < t$ is a birth-death pair if the alternating sum of ranks of the four lower left minors is positive. *Right:* the pair is shallow if furthermore row s and column t to the left and below the common entry are zero.

the lower left minors satisfy $r(u,v) = r(u,t) = r(s,v) = r(s,t) - 1$, so Lemma 3.3 implies that $s < t$ is also a birth-death pair; that is: $\mathrm{SH}(f) \subseteq \mathrm{BD}(f)$.

Another important property is that there are shallow pairs as long as there are birth-death pairs. More formally: $\mathrm{SH}(f) = \emptyset \implies \mathrm{BD}(f) = \emptyset$. In particular, the first death-giving cell and its last facet define a shallow pair. By symmetry so does the last birth-giving cell and its first cofacet. The cancellation of a shallow pair has a rather benign effect on the filter. Specifically, the canceled pair is the only one to disappear from the shallow pairs as well as from the birth-death pairs. Note however, that the operation may remove obstacles for birth-death pairs that were non-shallow before and become shallow after the cancellation.

▶ **Theorem 3.4** (Canceling a Shallow Pair)**.** *Let $f\colon X \to \mathbb{R}$ be a filter on a Lefschetz complex, $(s,t) \in X \times X$ a shallow pair, and $f'\colon X' \to \mathbb{R}$ the filter on the quotient after canceling (s,t). Then $\mathrm{SH}(f') \supseteq \mathrm{SH}(f) \setminus \{(s,t)\}$ and $\mathrm{BD}(f') = \mathrm{BD}(f) \setminus \{(s,t)\}$.*

Proof. We consider the boundary matrix of X whose rows and columns are ordered by the filter values, and the effect of a cancellation on it, as illustrated in Figure 2. To see the claim about the shallow pairs, recall that row s and column t to the left and below $\Delta[s,t]$ are zero. If we cancel (s,t) with column operations, we add column t to column y iff $\Delta[s,y] = 1$. But since (s,t) is shallow, this implies $f(t) < f(y)$, and assuming (x,y) is another shallow pair, this implies $f(s) < f(x)$. But then this column operation does not effect the portion of column y below $\Delta[x,y]$, so (x,y) remains shallow, as claimed.

To see the claim about the birth-death pairs, we note that adding column t to a column y to its right does not change the rank of any lower left minor of Δ. Since (s,t) is shallow, all column operations implementing the cancellation of (s,t) are of this kind, so Lemma 3.3 implies that all birth-death pairs remain, and no new ones get created. Thereafter (s,t) disappears when the rows and columns that correspond to s and t get removed. ◀

3.3 Shallow Orders

Theorem 3.4 motivates us to repeatedly cancel a shallow pair until there is none left. As mentioned in Section 3.2, there is a shallow pair as long as there are birth-death pairs. Since the cancellation does not change the other birth-death pairs, this implies that the iteration visits all birth-death pairs in an order such that each pair is shallow at the time of its cancellation. We use $[n]$ as a short-form for $\{1, 2, \ldots, n\}$.

▶ **Definition 3.5** (Shallow Orders). *Let $f \colon X \to \mathbb{R}$ be a filter on a Lefschetz complex, and $n = \#\mathrm{BD}(f)$ its number of birth-death pairs. A* shallow order *is a bijection $\Phi \colon [n] \to \mathrm{BD}(f)$ such that $\varphi_i = \Phi(i)$ is shallow after canceling φ_1 to φ_{i-1}, for each $1 \leq i \leq n$.*

Two particular shallow orders will be instrumental in the study of the dependencies between birth-death points, resp. their cancellations. To introduce them, we write φ_i° and $\varphi_i^\times$ for the birth-giving and death-giving cells of a birth-death pair φ_i. The first such special order prefers late births over early births, while the second prefers early deaths over late deaths:

$$\mathrm{A} \colon [n] \to \mathrm{BD}(f) \text{ such that } f(\alpha_i^\circ) > f(\alpha_{i+1}^\circ) \text{ for } 1 \leq i < n; \tag{3}$$

$$\Omega \colon [n] \to \mathrm{BD}(f) \text{ such that } f(\omega_i^\times) < f(\omega_{i+1}^\times) \text{ for } 1 \leq i < n, \tag{4}$$

in which A and Ω are bijections, and we write $\alpha_i = \mathrm{A}(i)$ and $\omega_i = \Omega(i)$. Note that α_1° is the last birth-giving cell in the filter, and $\alpha_1^\times$ is its first coface, which implies that α_1 is shallow. After canceling α_1, α_2 is shallow, etc., so A is indeed a shallow order. Symmetrically, $\omega_1^\times$ is the first death-giving cell, and ω_1° is its last facet, which again implies that ω_1 is shallow. By canceling ω_1 and iterating, we conclude that Ω is also a shallow order.

Example. We continue the example from Section 3.1 considering the 16 minima and maxima of the function on the circle shown in Figure 1. They form 7 birth-death pairs, with one minimum and one maximum unpaired. Following the two special shallow orders, we get

$$\mathrm{A}([7]) = ((\mathtt{h},\mathtt{G}),(\mathtt{e},\mathtt{E}),(\mathtt{d},\mathtt{C}),(\mathtt{f},\mathtt{F}),(\mathtt{b},\mathtt{B}),(\mathtt{c},\mathtt{D}),(\mathtt{g},\mathtt{H})); \tag{5}$$

$$\Omega([7]) = ((\mathtt{b},\mathtt{B}),(\mathtt{d},\mathtt{C}),(\mathtt{f},\mathtt{F}),(\mathtt{e},\mathtt{E}),(\mathtt{c},\mathtt{D}),(\mathtt{h},\mathtt{G}),(\mathtt{g},\mathtt{H})), \tag{6}$$

in which we write the elements of the images in sequence from 1 to 7. Note that both orders are linear extensions of the poset displayed in the lower right panel of Figure 1, a property we will explore next.

4 The Depth Poset

This section introduces the main new concept of this paper, the depth poset of a filter, which is a formalization of the dependencies between the birth-death pairs, respectively their cancellations. After defining the poset and proving some of its pertinent properties, we explain how to construct it, and finally establish the correctness of the algorithm.

4.1 Partial Order on Birth-death Pairs

A shallow order is a total order on the birth-death pairs or, equivalently, a complete graph with vertices $\mathrm{BD}(f)$ whose edges are directed in an acyclic manner. The intersection of two such graphs corresponds to a partial order such that both total orders are linear extensions of the poset. We apply this construction to the set of all shallow orders of a filter.

▶ **Definition 4.1** (Depth Poset). *Letting $f \colon X \to \mathbb{R}$ be a filter on a Lefschetz complex, the* depth poset, *denoted* $\mathrm{Depth}(f)$, *is the intersection of all shallow orders on* $\mathrm{BD}(f)$.

Its full name would be the *depth poset of canceling shallow birth-death pairs in a Lefschetz complex.* By definition, it is the largest partial order on $\mathrm{BD}(f)$ such that every shallow order is a linear extension. Note that $f(\varphi^\circ) > f(\psi^\circ)$ if φ precedes ψ in A, and $f(\varphi^\times) < f(\psi^\times)$ if φ precedes ψ in Ω. Since A and Ω are particular shallow orders, $(\varphi, \psi) \in \mathrm{Depth}(f)$ implies that φ precedes ψ in both, so the pairs are necessarily nested:

▶ **Proposition 4.2.** *Let $f\colon X \to \mathbb{R}$ be a filter on a Lefschetz complex, and φ, ψ two pairs in* $\mathrm{BD}(f)$*. Then $f(\psi^\circ) < f(\varphi^\circ) < f(\varphi^\times) < f(\psi^\times)$ whenever* $(\varphi, \psi) \in \mathrm{Depth}(f)$*.*

Observe that this implies that an ordering of the birth-death pairs by the difference between birth and death is necessarily a linear extension of the poset. We thus introduce

$$\Pi\colon [n] \to \mathrm{BD}(f) \;\text{ such that }\; f(\pi_i^\times) - f(\pi_i^\circ) \le f(\pi_{i+1}^\times) - f(\pi_{i+1}^\circ) \text{ for } 1 \le i < n, \qquad (7)$$

in which Π is a bijection and we write $\pi_i = \Pi(i)$. Note that Π orders the birth-death pairs by persistence. Proposition 4.2 thus implies that the pairs can be canceled in this sequence without otherwise affecting the filter:

▶ **Corollary 4.3.** *Let $f\colon X \to \mathbb{R}$ be a filter on a Lefschetz complex, and Π the ordering of the birth-death pairs by persistence. Then Π is a shallow order of f.*

Example. We note that the depth poset of the function on the circle in Figure 1 is consistent with the merge tree of the function; see e.g. [20]. There are however filtered graphs (1-dimensional Lefschetz complexes) with isomorphic merge trees for which the depth posets display different dependencies; see Figure 5. There are also examples of filtered graphs that have different merge trees but isomorphic depth posets (not shown).

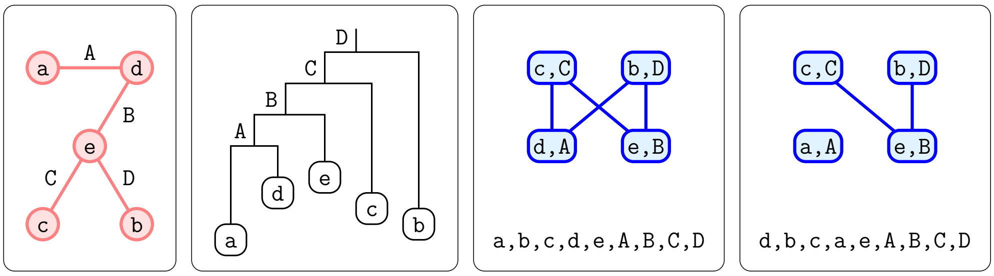

Figure 5: *From left to right:* a filtered graph, its merge tree (dendogram), its depth poset above the nodes and arcs ordered according to the filter, and the depth poset after swapping nodes `a` and `d` in the filter. The swap only causes nodes `a` and `d` to trade places in the dendogram, which does not affect the structure of the merge tree. In contrast, it shrinks the depth poset from four to two relations.

4.2 Transposing and Canceling Shallow Pairs

The conclusion that all linear extensions of the depth poset are shallow orders is not immediate since the definition of this poset is indirect. We therefore go slow and first establish some basic properties. Let $\Phi\colon [n] \to \mathrm{BD}(f)$ be a total order on the birth-death pairs of a filter. A *transposition* at positions $1 \le k, k+1 \le n$ produces another total order in which $\Phi(k)$ and $\Phi(k+1)$ are swapped. We are primarily interested in transpositions that swap shallow pairs.

▶ **Lemma 4.4.** *Let $f\colon X \to \mathbb{R}$ be a filter on a Lefschetz complex, and $\Phi, \Psi\colon [n] \to \mathrm{BD}(f)$ total orders that differ by the transposition at positions $1 \le k, k+1 \le n$. If Φ is a shallow order, and after canceling the first $k-1$ pairs, $\Phi(k+1)$ is a shallow pair, then Ψ is also a shallow order, and the quotients after canceling the first i pairs of Φ and Ψ, respectively, are the same for all $i \ne k$.*

Proof. The claim about the quotients is trivially true for $i < k$. For the next step, assume $i = k+1$ and write $\varphi = \Phi(k)$ and $\psi = \Phi(k+1)$. After canceling the first $k-1$ pairs, φ is shallow because Φ is a shallow order, and ψ is shallow by assumption. We transpose the

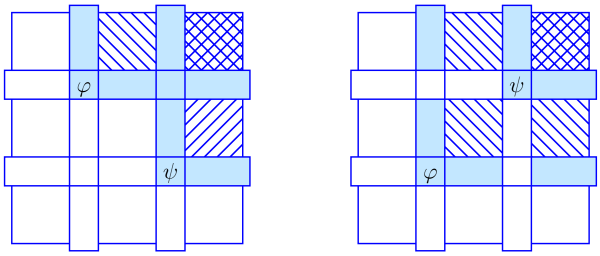

Figure 6: Two shallow pairs with possibly non-zero entries in the *shaded* portions of their rows and columns. On the *left*, the two pairs have either disjoint or incomparable intervals: $f(\varphi^\circ) < f(\psi^\circ)$ and $f(\varphi^\times) < f(\psi^\times)$, and on the *right*, their intervals are nested: $f(\psi^\circ) < f(\varphi^\circ) < f(\varphi^\times) < f(\psi^\times)$.

two pairs and argue that the swap does not affect the boundary matrix or, equivalently, the quotient after the $k+1$ cancellations. To see this, we write $\text{pvt}(\varphi)$ for the entry common to row φ° and column $\varphi^\times$, and note that it suffices to look at the entries to the upper right of $\text{pvt}(\varphi)$ and $\text{pvt}(\psi)$: the cross-hatched regions in Figure 6. Let t be the column of such an entry. Setting $\ell = \Delta_{k-1}[\varphi^\circ, t] + \Delta_{k-1}[\varphi^\circ, \psi^\times]$ and $m = \Delta_{k-1}[\psi^\circ, t] + \Delta_{k-1}[\psi^\circ, \varphi^\times]$, the effect of the two cancellations is adding ℓ times column $\varphi^\times$ and m times column $\psi^\times$ to column t. This is clear if the respective second terms are zero. The only other case is when $\Delta_{k-1}[\varphi^\circ, \psi^\times] = 1$, which may happen in the configuration illustrated on the left in Figure 6. If we first cancel ψ, then this changes the parity of $\Delta_{k-1}[\varphi^\circ, t]$, and if we first cancel φ, then we add column $\varphi^\times$ to column $\psi^\times$ before possibly adding it to column t. Either way, the effect is the same. Since ℓ and m are independent of the order in which φ and ψ are canceled, the quotients after canceling $i = k+1$ pairs in Φ and Ψ agree. For trivial reasons, the quotients therefore also agree after canceling $i > k+1$ pairs each, which implies that Ψ is also a shallow order, as claimed. ◀

Given a finite poset, it is not difficult to impose an acyclic relation on its linear extensions, such that two related extensions differ by a single transposition, and there is only a single maximum. We prove a similar result for the shallow orders, where we face the difficulty that we do not yet have a poset whose linear extensions are exactly the shallow orders.

▶ **Lemma 4.5.** *Let $f\colon X \to \mathbb{R}$ be a filter on a Lefschetz complex, and $\Phi, \Psi\colon [n] \to \text{BD}(f)$ two shallow orders. Then there is a sequence of shallow orders, $\Phi = \Phi_0, \Phi_1, \ldots, \Phi_m = \Psi$ such that Φ_{k-1}, Φ_k differ by a single transposition, for any $1 \leq k \leq m$.*

Proof. We fix a shallow order, which we construct iteratively by canceling all shallow pairs. Indeed, which birth-death pairs are shallow depends solely on $f_0 = f$. After canceling these shallow pairs, we get a filter $f_1\colon X_1 \to \mathbb{R}$ which, by Lemma 4.4, does not depend on the sequence in which we cancel these pairs. Next, we cancel the shallow pairs of f_1 to get $f_2\colon X_2 \to \mathbb{R}$, etc. Let $\Xi\colon [n] \to \text{BD}(f)$ be the sequence in which the pairs are canceled, and write $\xi_i = \Xi(i)$ for the i-th pair. By construction, Ξ is a shallow order of f.

Writing $\varphi_i = \Phi(i)$, we construct a sequence of transpositions that transform Φ into Ξ. In each iteration, we let ξ_i be the first pair in which Ξ differs from Φ, set $\varphi_k = \xi_i$, and move φ_k forward into i-th position using transpositions. By Lemma 4.4, each transposition produces a new shallow order, provided the two pairs are shallow prior to their transposition and after canceling all preceding pairs. But this is clear because the second of the two pairs is ξ_i, which is shallow after canceling the first $i-1$ pairs in Ξ. By construction, these $i-1$ pairs are also the first $i-1$ predecessors in Φ. We thus get a sequence of transpositions that transform Φ into Ξ, while each step preserves the property of the linear extension being shallow. Similarly, we construct such a sequence for Ψ and Ξ, and append its reverse to get a sequence of transpositions that transforms Φ into Ψ, as required. ◀

Call a terminal sequence of pairs in a shallow order a *suffix*, and the initial remainder the *complementary prefix*. We use the last two lemmas to show that the quotient after canceling all pairs in the prefix does not depend on the order in which the pairs are canceled.

▶ **Lemma 4.6.** *Let $f\colon X \to \mathbb{R}$ be a filter on a Lefschetz complex, and $\Phi, \Psi\colon [n] \to \mathrm{BD}(f)$ two shallow orders that share a common suffix. Then $X' = X''$, in which $f'\colon X' \to \mathbb{R}$ and $f''\colon X'' \to \mathbb{R}$ are the filters on the quotients after canceling the pairs in the complementary prefix of Φ and Ψ, respectively. Furthermore, the common depth poset of f' and f'' is the depth poset of f restricted to $\mathrm{BD}(f') = \mathrm{BD}(f'')$.*

Proof. By Lemma 4.5, we can transform the prefix of Φ into the prefix of Ψ by a sequence of transpositions that leaves the common suffix untouched. By Lemma 4.4, each such transposition preserves the quotients after canceling the transposed pair and all their predecessors. This implies that the quotient after canceling all pairs in the prefix remains constant throughout the sequence of transpositions. The claim about the depth poset follows because the shallow orders of $f' = f''$ are exactly the suffixes of the shallow orders of f. ◀

It will also be useful to have the following claim about the preservation of the row of the last birth-giving cell and the column of the first death-giving cell.

▶ **Lemma 4.7.** *Let $f\colon X \to \mathbb{R}$ be a filter on a Lefschetz complex, $\alpha_1 = \mathrm{A}^{-1}(1)$ and $\omega_1 = \Omega^{-1}(1)$ the respective leading pairs of the two special shallow orders, $\psi \neq \alpha_1, \omega_1$ a birth-death pair, and $\gamma \neq \psi$ a shallow such pair. Then the cancellation of γ preserves $\Delta[\alpha_1^\circ, \psi^\times]$ and $\Delta[\psi^\circ, \omega_1^\times]$, with $\gamma \neq \alpha_1$ in the first case and $\gamma \neq \omega_1$ in the second case.*

Proof. The two claims are symmetric, so it suffices to prove the first. To have an effect on the entries in row α_1°, it is necessary that $\Delta[\alpha_1^\circ, \gamma^\times] = 1$. But since α_1 and γ are both shallow, this is only possible if $f(\alpha_1^\circ) < f(\gamma^\circ)$, which contradicts $\alpha_1 = \mathrm{A}^{-1}(1)$. ◀

4.3 Lazy Reduction with Clearing

To construct the depth poset, we use variants of the standard matrix reduction algorithm for persistent homology; see e.g. [8, Chapter VII]. We begin with the variant of the column reduction algorithm, which differs from the classic algorithm in two ways. From [7] it borrows the idea that the columns can be reduced by taking the leftmost non-zero entries (pivots) in the rows from bottom to top. These pivots correspond to the birth-death pairs and their ordering prefers late over early births; compare with the first special shallow order, A, defined in (3). Note, however, that the birth-death pairs do not have to be known ahead of time as they are implicitly detected by the algorithm. From [3] it borrows the idea that rows and columns can be cleared after the corresponding pivot has been established. Indeed, after canceling two cells, we delete the corresponding rows and columns to get the boundary matrix of the quotient. At the same time, we collect the relations in an initially empty list; see Algorithm 1. By choice of the pivot, each pair (s, t) is shallow when it is visited, and the ensuing column operations effectively cancel s and t; see Definition 2.3 and Figure 2. By prioritizing late births, Algorithm 1 visits the pairs according to the first special shallow order and thus computes A. Letting Δ_i' be the matrix Δ' after i iterations of Algorithm 1, this is therefore the boundary matrix of the quotient after canceling α_1 through α_i. There are shallow pairs as long as there are birth-death pairs, so $\Delta_i' \neq 0$, for all $i < n$, and $\Delta_n' = 0$. Hence, Algorithm 1 halts after $n = \#\mathrm{BD}(f)$ iterations.

Algorithm 1 Bottom to Top Column Reduction

1: $\Delta' = \Delta$; $B' = \emptyset$; $i = 0$;
2: **while** $\Delta' \neq 0$ **do** $i = i + 1$;
3:     let $\Delta'[s,t]$ be leftmost non-zero entry in last non-zero row; $\alpha_i = (s,t)$;
4:     **while** $\exists y > t$ such that $\Delta'[s,y] = 1$ **do**
5:         add column t to column y in Δ'; append (t,y) to B'
6:     **end while**;
7:     delete rows s and t and columns s and t from Δ'
8: **end while**.

Algorithm 2 Left to Right Row Reduction

1: $\Delta'' = \Delta$; $B'' = \emptyset$; $j = 0$;
2: **while** $\Delta'' \neq 0$ **do** $j = j + 1$;
3:     $\Delta''[s,t]$ is lowest non-zero entry in first non-zero column; $\omega_j = (s,t)$;
4:     **while** $\exists x < s$ such that $\Delta''[x,t] = 1$ **do**
5:         add row s to row x in Δ''; append (s,x) to B'';
6:     **end while**;
7:     delete rows s and t and columns s and t from Δ''
8: **end while**.

Symmetrically, Algorithm 2 computes Ω while reducing the boundary matrix with row operations. Letting Δ''_j be the matrix Δ'' after j iterations of Algorithm 2, it is the boundary matrix of the quotient after canceling ω_1 through ω_j, and Algorithm 2 also halts after n iterations. We state this for later reference:

▶ **Lemma 4.8.** *Let $f\colon X \to \mathbb{R}$ be a filter on a Lefschetz complex, and Δ'_i, Δ''_j as defined above. Then Δ'_i is the boundary matrix of the quotient after canceling $\alpha_1, \alpha_2, \ldots, \alpha_i$, and Δ''_j is the boundary matrix of the quotient after canceling $\omega_1, \omega_2, \ldots, \omega_j$.*

4.4 Relations from Book-keeping

The relations collected by the two algorithms do not contradict each other: if $(\varphi^\times, \psi^\times)$ is in the transitive closure of B', then $f(\psi^\circ) < f(\varphi^\circ) < f(\varphi^\times) < f(\psi^\times)$, and if $(\psi^\circ, \varphi^\circ)$ is in the transitive closure of B'', then we get the reverse of these inequalities. This excludes their co-occurrence, so it makes sense to define the transitive closure of the union:

$$R(f) = \text{closure}\{(\varphi,\psi) \mid (\varphi^\times, \psi^\times) \in B' \text{ or } (\varphi^\circ, \psi^\circ) \in B''\}. \tag{8}$$

We claim that $R(f)$ is in fact the depth poset of f. While this is plausible, it is not obvious and requires a proof.

▶ **Theorem 4.9.** *Let $f\colon X \to \mathbb{R}$ be a filter on a Lefschetz complex. Then* $\mathrm{Depth}(f) = R(f)$.

Proof. We first prove $R(f) \subseteq \mathrm{Depth}(f)$. It suffices to argue the containment for the relations supplied by B', with the argument for B'' being symmetric. Let therefore $(\varphi,\psi) \in R(f)$ and recall that $(\varphi^\circ, \psi^\times) \in B'$ iff $\Delta'_i[\varphi^\circ, \psi^\times] = 1$; see Lines 4 and 5 of Algorithm 1. Set $i + 1 = \mathrm{A}^{-1}(\varphi)$ and write $f'_i\colon X'_i \to \mathbb{R}$ for the filter after canceling $\alpha_1, \alpha_2, \ldots, \alpha_i$. By Lemma 4.6, $(\varphi,\psi) \in \mathrm{Depth}(f)$ iff $(\varphi,\psi) \in \mathrm{Depth}(f'_i)$, so we can focus on the situation after canceling the first i pairs in A. The pivots are processed from bottom to top, which implies $f(\varphi^\circ) > f(\psi^\circ)$; that is: $\mathrm{pvt}(\varphi)$ is below $\mathrm{pvt}(\psi)$, as in Figure 7 on the left. The question is whether there is any sequence of pairs of f_i'—excluding φ—whose cancellation makes ψ a shallow pair, so it can be canceled before φ. For such a sequence to exist, there must be a pair, γ, whose cancellation changes $\Delta'[\varphi^\circ, \psi^\times]$ from 1 to 0. However, since φ is the leading pair in the first special shallow order of f_i', this is prohibited by Lemma 4.7.

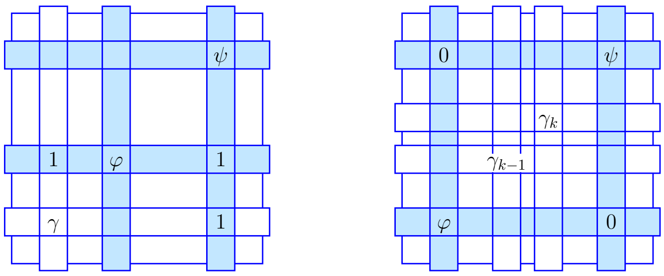

Figure 7: The matrix on the *left* illustrates first part of the proof: to cancel γ, we would add its column to the column of ψ, with the effect that $\Delta'[\varphi^\circ, \psi^\times]$ changes from 1 to 0. But such γ does not exist after canceling all pairs below $\mathrm{pvt}(\varphi)$. The matrix on the *right* illustrates the second part of the proof: after transposing γ_{k-1} and γ_k, we are one step closer to a contradiction.

We second prove $\mathrm{Depth}(f) \subseteq R(f)$. It suffices to argue the containment for pairs $(\varphi, \psi) \in \mathrm{Depth}(f)$ that are not implied by transitivity. Set $i+1 = \mathrm{A}^{-1}(\varphi)$ and $j+1 = \Omega^{-1}(\varphi)$, write $f_i' \colon X_i' \to \mathbb{R}$ and Δ_i' after canceling the prefix of length i in A, $f_j'' \colon X_j'' \to \mathbb{R}$ and Δ_j'' after canceling the prefix of length j in Ω, and $f_{ij} \colon X_{ij} \to \mathbb{R}$ and Δ_{ij} after canceling the pairs in both prefixes. By Lemmas 4.6 and 4.7, we have $\Delta_{ij}[\varphi^\circ, \psi^\times] = \Delta_i'[\varphi^\circ, \psi^\times]$ and $\Delta_{ij}[\psi^\circ, \varphi^\times] = \Delta_j''[\psi^\circ, \varphi^\times]$. If either of these two entries is 1, then $(\varphi, \psi) \in R(f)$ and we are done. Hence, assume $\Delta_{ij}[\varphi^\circ, \psi^\times] = \Delta_{ij}[\psi^\circ, \varphi^\times] = 0$, as in Figure 7 on the right. Set $n_{ij} = \#\mathrm{BD}(f_{ij})$, and let $\Phi \colon [n_{ij}] \to \mathrm{BD}(f_{ij})$ be a linear extension of $\mathrm{Depth}(f_{ij})$ that minimizes $m = \Phi^{-1}(\psi) - \Phi^{-1}(\varphi)$. We already established $R(f_{ij}) \subseteq \mathrm{Depth}(f_{ij})$, so Φ is also a linear extension of $R(f_{ij})$. If $m = 1$, then we can transpose φ and ψ and thus contradict $(\varphi, \psi) \in \mathrm{Depth}(f_{ij})$. So assume $m \geq 2$ and write $\varphi = \gamma_0, \gamma_1, \ldots, \gamma_m = \psi$ for the relevant subsequence. Since (φ, ψ) is not implied by transitivity, there is no chain of two or more relations that connects φ to ψ in the depth poset. Hence, there is an index $1 \leq k \leq m$ such that $(\gamma_{k-1}, \gamma_k) \notin \mathrm{Depth}(f_{ij})$, and since $R(f_{ij}) \subseteq \mathrm{Depth}(f_{ij})$, we also have $(\gamma_{k-1}, \gamma_k) \notin R(f_{ij})$. If (φ, γ_{k-1}) or (γ_k, ψ) is a relation in $\mathrm{Depth}(f_{ij})$, then we transpose γ_{k-1} and γ_k and get a new linear extension of $\mathrm{Depth}(f_{ij})$ and $R(f_{ij})$. There is necessarily at least one pair to transpose, so if we iterate, the predecessors of ψ migrate to the left, and the successors of φ migrate to the right. Eventually, we get a transposition that involves $\varphi = \gamma_0$ or $\psi = \gamma_m$, and either way, we get a linear extension that contradicts our choice of Φ. ◀

By definition of the depth poset, every shallow order of f is a linear extension of $\mathrm{Depth}(f)$. Using Theorem 4.9, it is not difficult to argue also the converse, which implies that the shallow orders and the linear extensions of the depth poset are indeed one and the same.

Observe that Algorithm 1 adds a column t to another column y only if the two cells satisfy $\dim t = \dim y$. Similarly, Algorithm 2 adds a row s to another row x only if $\dim s = \dim x$. By Theorem 4.9, this implies that every relation in $\mathrm{Depth}(f)$ is between birth-death pairs of the same pair of dimensions. Hence, the depth poset is a disjoint union of one poset per pair of consecutive dimensions. This motivates us to define $\mathrm{BD}_p(f) \subseteq \mathrm{BD}(f)$ as the birth death-pairs $\varphi = (\varphi^\circ, \varphi^\times)$ with $\dim \varphi^\circ = \dim \varphi^\times - 1 = p$, and similarly restrict the depth poset by defining $\mathrm{Depth}_p(f) = \mathrm{Depth}(f) \cap (\mathrm{BD}_p(f) \times \mathrm{BD}_p(f))$.

▶ **Corollary 4.10.** *Let $f\colon X \to \mathbb{R}$ be a filter on a Lefschetz complex of dimension d. Then* $\mathrm{Depth}(f) = \mathrm{Depth}_0(f) \sqcup \mathrm{Depth}_1(f) \sqcup \ldots \sqcup \mathrm{Depth}_{d-1}(f)$.

This corollary is relevant if we annotate a persistence diagram with arcs that connect points representing birth-death pairs related to each other in the depth poset; see Figure 8 for the 0-dimensional persistence diagram of a 1-dimensional function. By Corollary 4.10, each such arc belongs to a unique dimension.

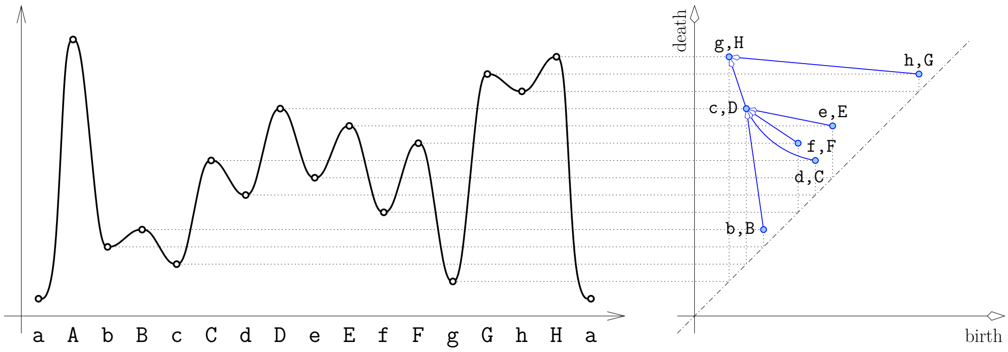

Figure 8: The function on the circle introduced in Figure 1 on the *left*, and its persistence diagram overlayed with the depth poset on its birth-death pairs on the *right*.

5 Discussion

The main contribution of this paper are the introduction of the depth poset—which records and organizes the dependencies between the cancellations of shallow birth-death pairs in a Lefschetz complex—and a proof that it can be constructed by a customized but otherwise straightforward matrix reduction algorithm. The novel structure raises a number of questions and opens opportunities for further work:

- It would be interesting to perform stochastic experiments to understand the statistical behavior of the depth poset. Are differences in the local structure of the relations helpful in detecting outliers or in the reduction of noise in sampled data?
- It would also be interesting to analyze the sensitivity of the depth poset to transpositions in the filter, as this may be correlated with the changing dynamics of the gradient flow; see [6, 15, 17] for related work in this direction.

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