

# The Poset of Cancellations in a Filtered Complex

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## 1 Abstract

Motivated by questions about simplification and topology optimization, we take a discrete approach toward the dependency of topology simplifying operations and the reachability of perfect Morse functions. Representing the function by a filter on a Lefschetz complex, and its (non-essential) topological features by the pairing of its cells via persistence, we simplify using combinatorially defined cancellations. The main new concept is the *depth poset* on these pairs, whose linear extensions are schedules of cancellations that trim the Lefschetz complex to its essential homology. One such linear extension is the cancellation of the pairs in the order of their persistence. An algorithm that constructs the depth poset in two passes of standard matrix reduction is given and proven correct.

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## 10 1 Introduction

The primary aim of this paper is to shed light on the general question of simplification while preserving topology or, more specifically, on the dependencies between the operations that locally simplify. Examples are cancellations of critical point pairs in a Morse function, and collapses of simplex pairs in a simplicial complex. Depending on the sequence, these operations may or may not succeed in producing a perfect Morse function or a single vertex complex. Another source of motivation is the optimization of topology. To relate the two problems, we may think of ‘simplifying’ a function on a domain, while ‘optimizing’ the topology of a sublevel set of that function. The target of the optimization may address topology directly (such as minimizing the Betti numbers under some constraints) or indirectly (such as maximizing the strength-to-weight ratio of a shape). Optimizing shapes for everyday use is important, so there is a discipline within engineering dedicated to this subject [5].

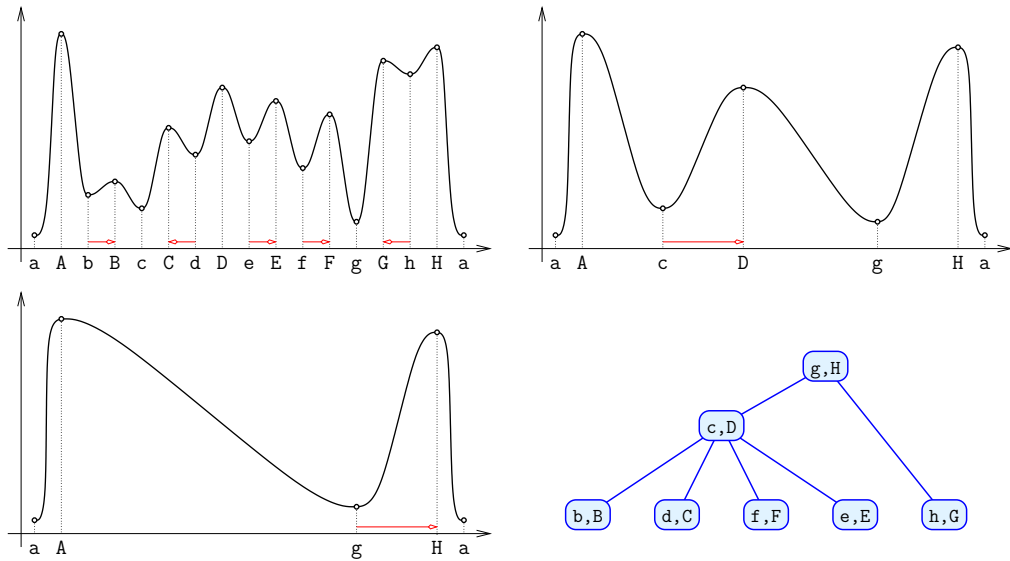
The approach to these problems taken in this paper<sup>1</sup> is discrete and based on *Lefschetz complexes* [14] to represent shapes or spaces, which are abstractions of the more geometric cellular complexes. In this context, a continuous function is replaced by a filter, which maps cells to real numbers satisfying the mild requirement that the faces of a cell receive values

<sup>1</sup> A subset of the results appeared in an earlier version of this paper [10]. The connection to concepts in discrete Morse theory [12], and extensions to combinatorial dynamics [17] will be reported elsewhere.



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31 smaller than the cell. The operations are *cancellations* of cell in pairs, and preferably in  
 32 *shallow pairs*—which were introduced under the name *apparent pairs* in [2]—as they preserve  
 33 the rest of the topological structure to the extent this is possible.<sup>2</sup> The dependence between  
 34 cancellations arises because pairs may or may not become shallow depending on which  
 35 shallow pairs are canceled in which sequence. These dependencies are captured by the *depth*  
 36 *poset*, which we construct using customized matrix reduction algorithms, and which may be  
 37 used to annotate the persistence diagram of the filter. Figure 1 shows an example in the  
 38 simplistic setting of a function on a circle: three rounds of cancellations of shallow min-max  
 39 pairs suffice to produce a function with a single min-max pair, and the poset at the lower  
 right presents all linear schedules of shallow cancellations.



■ Figure 1: *Upper left*: a generic smooth function with 8 minima and 8 maxima on a circle. *Upper right*: simplified version of the function after canceling all 5 shallow min-max pairs, which are indicated by red arrows. The 5 cancellations turn a former non-shallow min-max pair shallow, whose cancellation leads to the further simplified version of the function at the *lower left*. The cancellation of the last birth-death pair, which is now shallow, produces a function with a single minimum and a single maximum (not shown). *Lower right*: the depth poset, whose relations express the dependencies between the cancellations: its linear extensions are sequences such that each pair is shallow at the time it is canceled.

40

41 There is related prior work on simplifying piecewise linear functions using the persistence  
 42 diagram to quantify distortion, which gives satisfying results for 2-manifolds but runs into  
 43 topological obstacles for 3-manifolds [1, 4]. The prior work on topological optimization most  
 44 directly related to this paper has focused on operations that move points in the persistence  
 45 diagram, which include cancellations [11, 18]. The customized matrix reduction we use to  
 46 construct the depth poset uses elements of the column and row reduction algorithms for  
 47 persistent homology described in [3, 7].

48 **Outline.** Section 2 explains Lefschetz complexes and cancellations. Section 3 introduces  
 49 shallow pairs as special birth-death pairs defined in persistent homology. Importantly, it  
 50 identifies two special total orders along which all cancellations are of shallow pairs. Section 4

24 <sup>2</sup> In the case of a 1-dimensional function, a min-max pair is shallow iff the max is the lower of the two  
 25 neighboring maxima of the min, and the min is the higher of the two neighboring minima of the max.  
 26 The structure of these pairs was recently exploited in adaptive sorting of lists [19].

51 defines the main new concept, the depth poset, proves some of its properties, and gives  
 52 a matrix reduction algorithm to construct it. One of the off-shots of this construction is  
 53 the insight that the order of the birth-death pairs by persistence also enjoys the property  
 54 mentioned for the two special total orders. Finally, Section 5 concludes the paper.

## 55 **2 Cancellations in Lefschetz Complexes**

56 We are interested in the dependence of the topological features of a function on a space  
 57 or, in the discrete setting studied in this paper, of a filter on a complex. To make this  
 58 concrete, we need to specify what we mean by a feature, and what family of complexes  
 59 and operations between them we consider. This section fixes the latter two variables to the  
 60 Lefschetz complexes and cancellations between them, while it leaves the discussion of the  
 61 features to the next section.

### 62 **2.1 Lefschetz Complexes**

63 We work with an abstraction of a geometric cellular complex, referred to as a Lefschetz  
 64 complex. It keeps track of the dimension of each cell and its incidences with cells of one lower  
 65 or higher dimension, but it does not worry about geometric details, such as how the cells are  
 66 attached to each other. To simplify its exposition, we use modulo-2 arithmetic throughout  
 67 this paper, which amounts to working with homology for coefficients in  $\mathbb{Z}/2\mathbb{Z}$ .

68 ► **Definition 2.1** (Lefschetz Complex). *A Lefschetz complex is a triplet  $(X, \dim, \Delta)$ , in which  
 69  $X$  is a finite set of elements called cells,  $\dim: X \rightarrow \mathbb{Z}$  maps each cell to its dimension, and  
 70  $\Delta: X \times X \rightarrow \{0, 1\}$  is a map such that  $\Delta(x, y) \neq 0$  only if  $\dim y = \dim x + 1$ , and*

$$71 \quad \sum_{y \in X} \Delta(x, y) \cdot \Delta(y, z) = 0 \quad (1)$$

72 *holds for all  $x, z \in X$ . If  $\Delta(x, y) = 1$ , we call  $x$  a facet of  $y$ , we call  $y$  a cofacet of  $x$ , and  
 73 we write  $x < y$  to denote this relation. The dimension of  $X$  is  $\dim X = \max_{x \in X} \dim x$ .*

74 We will sometimes shorten the notation and refer to  $X$  as a Lefschetz complex. Using  
 75 Equation (1), we associate with  $X$  a chain complex and homology, following the same  
 76 standard scheme as for cellular complexes. Reusing the notation  $\Delta$  for the associated  
 77 boundary matrix, we observe that  $\Delta[x, y] = \Delta(x, y)$ . Whenever convenient, we split  $\Delta$  into  
 78 the boundary matrices dedicated to individual dimensions, with  $\Delta_p$  recording the incidences  
 79 between cells of dimension  $p$  and  $p - 1$ .

80 The abstraction of a cellular complex to its Lefschetz complex is with controlled loss of  
 81 information. Beyond the geometric details, we also lose information about the homotopy type.  
 82 An example is the 3-sphere, which may be represented by the Lefschetz complex consisting  
 83 of two isolated cells, one of dimension 0 and the other of dimension 3. The same Lefschetz  
 84 complex represents the Poincaré homology 3-sphere, which has isomorphic homology groups  
 85 but a different homotopy type than the 3-sphere [13].

86 A simplicial complex and its barycentric subdivision have identical underlying spaces and  
 87 therefore isomorphic homology groups. Abstractly, the barycentric subdivision corresponds to  
 88 the *order complex* of the face poset of the simplicial complex. This observation generalizes to  
 89 *regular complexes*, whose cells are topological balls attached to each other via homeomorphic  
 90 gluing maps, but not necessarily to cellular complexes with more complicated gluing maps.  
 91 Alternatively, we can consider the free chain complex defined by the Lefschetz complex and  
 92 define its homology from the corresponding cycle and boundary groups. While the thus

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93 obtained homology of a Lefschetz complex is generally different from the singular homology  
 94 of the corresponding order complex, the two agree when the Lefschetz complex represents a  
 95 regular complex. Indeed the following is a corollary of a theorem by McCord from 1966.

96 ► **Theorem 2.2** (McCord [16]). *If  $X$  is a regular complex, then the homology of the free*  
 97 *chain complex defined by its Lefschetz complex is isomorphic to the singular homology of  $X$ .*

98 The following two subsections give the reasons we will work with the homology of the  
 99 free chain complex, also for cases in which the Lefschetz complex does not correspond to a  
 100 regular complex, such as the ones in Figure 3.

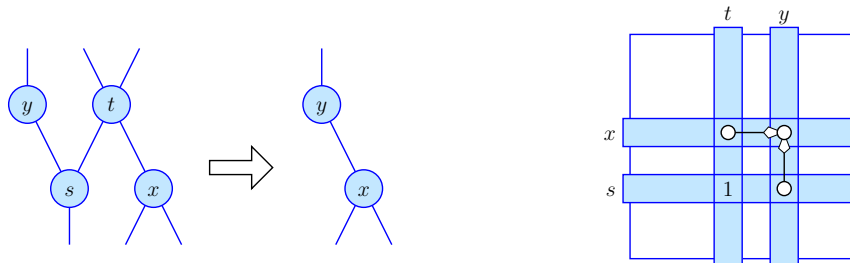
### 101 2.2 Cancellations

102 Intuitively, a cancellation in a complex is like a collapse, except that it can also happen  
 103 inside and thus away from the boundary. Such an “interior collapse” has consequences, as  
 104 it distorts cells and may turn a regular complex into one in which the gluing maps are no  
 105 longer homeomorphic. We cope with these consequences by ignoring them on the account of  
 106 the more abstract Lefschetz complex.

107 ► **Definition 2.3** (Cancellation). *Let  $(X, \dim, \Delta)$  be a Lefschetz complex and  $s < t$  both in  $X$ .*  
 108 *The cancellation of the pair removes both cells and updates the incidence relation accordingly.*  
 109 *Specifically, it sets  $X' = X \setminus \{s, t\}$ ,  $\dim' = \dim|_{X'}$ , and  $\Delta' : X' \times X' \rightarrow \{0, 1\}$  such that*

$$110 \quad \Delta'(x, y) = \Delta(x, y) + \Delta(s, y) \cdot \Delta(x, t), \quad (2)$$

111 *for all  $x, y \in X'$ . We refer to  $(X', \dim', \Delta')$  as the quotient after canceling  $s$  and  $t$ .*



■ Figure 2: The effect of canceling  $s < t$  on the Lefschetz complex on the *left* and the boundary matrix on the *right*. If in addition  $x$  were also incident to  $y$ , then the cancellation would removed this incidence, leaving  $y$  without child and  $x$  without parent (not shown).

112 Figure 2 illustrates the effect of canceling  $s < t$ . In particular, the cancellation adds column  
 113  $t$  to every other column  $y$  for which  $s < y$  or, alternatively, it adds row  $s$  to every other row  
 114  $x$  for which  $x < t$ . After either the column or the row operations, the cancellation removes  
 115 rows  $s$  and  $t$  as well as columns  $s$  and  $t$  from the matrix. It is not difficult to see that the  
 116 quotient is again a Lefschetz complex. More importantly, the cancellation preserves the  
 117 homology of the complex, since it translates into row or column operations that preserve the  
 118 ranks of the individual boundary matrices. We state this for later reference.

119 ► **Proposition 2.4.** *A Lefschetz complex and its quotient after canceling a facet-cofacet pair*  
 120 *have isomorphic homology groups.*

126 *Example.* To show that cancellations in a Lefschetz complex are more powerful than collapses,  
 127 we illustrate how they remove a Dunce hat [21] attached to one end of a cylinder in Figure 3.

128 To get the ranks of the homology groups, we count the generators of the cycle and boundary  
 129 groups; see Table 1. As predicted by Proposition 2.4, the cancellation does not affect the  
 ranks of the homology groups.

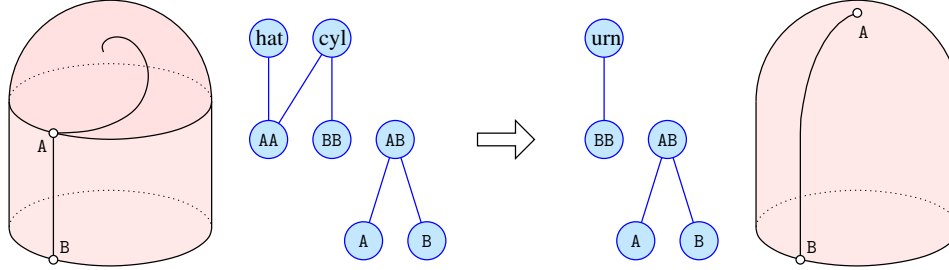


Figure 3: Far left: a cylinder cut along the edge AB, which connects the points A and B on its two boundary circles (represented by the edges AA and BB), and (an artistic sketch of) a Dunce hat attached to AA three times. Far right: after canceling the Dunce hat and AA, we get an upside-down urn cut along the edge connecting A (to which the Dunce hat contracted) to B. In the middle: the Lefschetz complexes before and after the cancellation of the Dunce hat.

	before			after		
	$Z_p$	$B_p$	$\beta_p$	$Z_p$	$B_p$	$\beta_p$
$p = 0$	A, B	A+B	1	A, B	A+B	1
$p = 1$	AA, BB	AA, BB	0	BB	BB	0
$p = 2$	$\emptyset$	$\emptyset$	0	$\emptyset$	$\emptyset$	0

Table 1: The generators of the cycle and boundary groups of the Lefschetz complexes in Figure 3. Recall that these complexes differ by canceling the Dunce hat at the top of the space on the left.

### 3 Shallow and Other Birth-death Pairs

In this section, we return to the notion of a topological feature, which for a filtered complex will be a birth-death pair of cells. We define them in a quick introduction to persistent homology and refer to [8] for more comprehensive background on this topic. Among the birth-death pairs, we will single out the simplest kind as shallow pairs, which we use to explore the dependence between all birth-death pairs of a given ordered complex.

#### 3.1 Persistent Homology

By a *filter* of a Lefschetz complex,  $X$ , we mean an injective function  $f: X \rightarrow \mathbb{R}$  such that  $f(x) < f(y)$  whenever  $x < y$ . Write  $X_b = f^{-1}(-\infty, b]$  for the sublevel set at  $b \in \mathbb{R}$ . By construction, every sublevel set of  $f$  is a Lefschetz complex, and we refer to the increasing sequence of distinct sublevel sets as the *filtration* induced by  $f$ . To describe how the homology changes as we move from one sublevel set to the next, we write  $[d]_b$  for the homology class of a cycle  $d \in Z(X_b)$ . Let  $a < b$  be consecutive values of  $f$ , so there are cells  $x, y \in X$  such that  $a = f(x)$ ,  $b = f(y)$ , and  $X_b = X_a \cup \{y\}$ . Since  $\partial y$  is a boundary in  $X_b$ ,  $[\partial y]_b = 0$ , and if  $[\partial y]_a \neq 0$ , then we say  $y$  gives death to a homology class. Otherwise, there is a chain  $c \in C(X_a)$  such that  $\partial c = \partial y$ . In this case,  $c + y$  is a cycle, and  $[c + y]_b \neq 0$  because  $X_b$  contains no cofacet of  $y$  yet, so  $c_y$  cannot be a boundary in  $X_b$ . We therefore say  $y$  gives birth to  $[c + y]_b$ . We write  $X^\circ$  and  $X^\times$  for the cells in  $X$  that give birth and death respectively. Every cell does either, so  $X^\circ \cap X^\times = \emptyset$  and  $X^\circ \cup X^\times = X$ .

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150 We need additional notions to associate births with deaths. First note that the homology  
151 class  $[c + y]_b$  given birth to by  $y$  is generally not unique. To fix this inconvenience, we observe  
152 that there is a unique chain  $c_y \in \mathcal{C}(X_a)$  such that  $\partial c_y = \partial y$  and  $c_y \subseteq X^\times$ . Clearly,  $y$  gives  
153 birth to the homology class of  $d_y = c_y + y$ , and we call  $d_y$  the *canonical cycle* associated with  
154  $y$ . Following the original persistence algorithm in [9], we use an inductive argument to define  
155 the pairing and simultaneously a set  $Y_a \subseteq X_a$ , which we will see contains all birth-giving cells  
156 that are not yet paired. Initially, both these sets are empty. For the inductive step, assume  
157 we have  $Y_a \subseteq X_a$ , and let  $b$  be the next value, after  $a$ , and  $y$  the next cell, with  $f(y) = b$ . If  
158  $y \in X^\circ$ , we set  $Y_b = Y_a \cup \{y\}$ . Otherwise  $y \in X^\times$ , which implies  $[d_y]_b \neq 0$ . There is a unique  
159 subset  $A \subseteq Y_a$  such that  $d' = \sum_{x \in A} d_x$  satisfies  $[d']_a = [\partial y]_a$ . We let  $z$  be the last cell in  $A$   
160 (the cell with maximum value), write  $\text{bth}(y) = z$ , and set  $Y_b = Y_a \setminus \{\text{bth}(y)\}$ . This defines  
161 an injective map,  $\text{bth}: X^\times \rightarrow X^\circ$ , but note that it is not necessarily bijective since there  
162 may be cells in  $X^\circ$  that never die. They represent the homology of  $X$ .

163 ► **Definition 3.1** (Birth-death Pairs). *Let  $f: X \rightarrow \mathbb{R}$  be a filter on a Lefschetz complex. Then*  
164  *$(s, t) \in X \times X$  is a birth-death pair of  $f$  if  $s = \text{bth}(t)$ .*

165 *Example.* Consider the function on the circle displayed in the upper left panel in Figure 1.  
166 Representing the function by a filter, we let each minimum be a vertex, whose filter value  
167 is the function value (height) of the minimum, and each maximum an edge, whose filter value  
168 is the height of the maximum. The birth-death pairs marked by arrows in the upper  
169 left panel are (b, B), (d, C), (e, E), (f, F), (h, G), and the remaining birth-death pairs marked  
170 by arrows in the upper right and the lower left panels are (c, D) and (g, H). As we will see  
171 shortly, the first five birth-death pairs are shallow, and the last two are not. The remaining  
172 two critical points, the minimum a and the maximum A, both give birth and are unpaired as  
173 they represent the homology of the circle (one component and one 1-cycle).

### 174 3.2 Shallow Pairs

175 A birth-death pair,  $(s, t) \in \text{BD}(f)$ , can be cancelled if  $s$  is a facet of  $t$  in the Lefschetz  
176 complex. To avoid that this cancellation affects other birth-death pairs, we limit ourselves to  
177 canceling only special such pairs.

178 ► **Definition 3.2** (Shallow Pairs). *Let  $f: X \rightarrow \mathbb{R}$  be a filter on a Lefschetz complex. A pair*  
179  *$(s, t) \in X \times X$  is shallow if  $s$  is the last facet of  $t$  and  $t$  is the first cofacet of  $s$  in the filter,*  
180 *and we write  $\text{SH}(f)$  for the set of shallow pairs of the filter.*

181 In other words,  $(s, t) \in \text{SH}(f)$  if  $f(x) \leq f(s)$  for all  $x < t$  and  $f(y) \geq f(t)$  for all  $y > s$ . We  
182 use the *ordered boundary matrix*—whose rows and columns are sorted by filter values—to  
183 recognize when  $s < t$  is a shallow pair, or just a birth-death pair. With reference to the left  
184 panel of Figure 4, we write  $r(s, t)$  for the rank of the lower left minor obtained by deleting all  
185 rows above row  $s$  and columns to the right of column  $t$  in the boundary matrix, and we let  $u$   
186 be the row right after (below) row  $s$  and  $v$  the column right before (to the left of) column  
187  $t$ . The following proposition is the Pairing Uniqueness Lemma in [6] restated for Lefschetz  
188 complexes:

189 ► **Lemma 3.3** (Cohen-Steiner et al. 2006). *Let  $f: X \rightarrow \mathbb{R}$  be a filter on a Lefschetz complex,*  
190 *and  $s, t \in X$  two of its cells. Then  $(s, t) \in \text{BD}(f)$  iff  $r(s, t) - r(s, v) - r(u, t) + r(u, v) > 0$ .*

191 Compare this with a shallow pair in which row  $s$  is zero to the left of column  $t$ , and column  
192  $t$  is zero below row  $s$ ; see the right panel of Figure 4. Assuming  $s < t$  is shallow, the ranks of

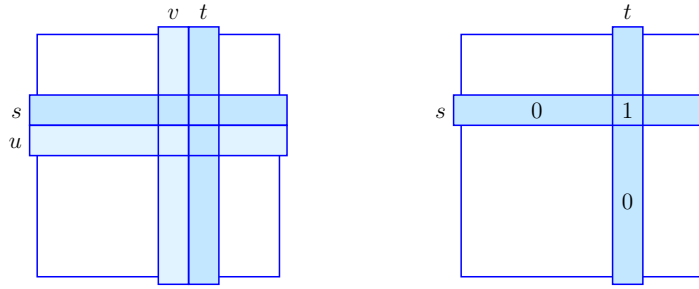


Figure 4: *Left*: the  $s < t$  is a birth-death pair if the alternating sum of ranks of the four lower left minors is positive. *Right*: the pair is shallow if furthermore row  $s$  and column  $t$  to the left and below the common entry are zero.

193 the lower left minors satisfy  $r(u, v) = r(u, t) = r(s, v) = r(s, t) - 1$ , so Lemma 3.3 implies  
 194 that  $s < t$  is also a birth-death pair; that is:  $\text{SH}(f) \subseteq \text{BD}(f)$ .

195 Another important property is that there are shallow pairs as long as there are birth-death  
 196 pairs. More formally:  $\text{SH}(f) = \emptyset \implies \text{BD}(f) = \emptyset$ . In particular, the first death-giving cell  
 197 and its last facet define a shallow pair. By symmetry so does the last birth-giving cell and  
 198 its first cofacet. The cancellation of a shallow pair has a rather benign effect on the filter.  
 199 Specifically, the canceled pair is the only one to disappear from the shallow pairs as well  
 200 as from the birth-death pairs. Note however, that the operation may remove obstacles for  
 201 birth-death pairs that were non-shallow before and become shallow after the cancellation.

202 ► **Theorem 3.4** (Canceling a Shallow Pair). *Let  $f: X \rightarrow \mathbb{R}$  be a filter on a Lefschetz complex,*  
 203  *$(s, t) \in X \times X$  a shallow pair, and  $f': X' \rightarrow \mathbb{R}$  the filter on the quotient after canceling  $(s, t)$ .*  
 204 *Then  $\text{SH}(f') \supseteq \text{SH}(f) \setminus \{(s, t)\}$  and  $\text{BD}(f') = \text{BD}(f) \setminus \{(s, t)\}$ .*

205 **Proof.** We consider the boundary matrix of  $X$  whose rows and columns are ordered by the  
 206 filter values, and the effect of a cancellation on it, as illustrated in Figure 2. To see the claim  
 207 about the shallow pairs, recall that row  $s$  and column  $t$  to the left and below  $\Delta[s, t]$  are  
 208 zero. If we cancel  $(s, t)$  with column operations, we add column  $t$  to column  $y$  iff  $\Delta[s, y] = 1$ .  
 209 But since  $(s, t)$  is shallow, this implies  $f(t) < f(y)$ , and assuming  $(x, y)$  is another shallow  
 210 pair, this implies  $f(s) < f(x)$ . But then this column operation does not effect the portion of  
 211 column  $y$  below  $\Delta[x, y]$ , so  $(x, y)$  remains shallow, as claimed.

212 To see the claim about the birth-death pairs, we note that adding column  $t$  to a column  
 213  $y$  to its right does not change the rank of any lower left minor of  $\Delta$ . Since  $(s, t)$  is shallow,  
 214 all column operations implementing the cancellation of  $(s, t)$  are of this kind, so Lemma 3.3  
 215 implies that all birth-death pairs remain, and no new ones get created. Thereafter  $(s, t)$   
 216 disappears when the rows and columns that correspond to  $s$  and  $t$  get removed. ◀

### 217 3.3 Shallow Orders

218 Theorem 3.4 motivates us to repeatedly cancel a shallow pair until there is none left. As  
 219 mentioned in Section 3.2, there is a shallow pair as long as there are birth-death pairs. Since  
 220 the cancellation does not change the other birth-death pairs, this implies that the iteration  
 221 visits all birth-death pairs in an order such that each pair is shallow at the time of its  
 222 cancellation. We use  $[n]$  as a short-form for  $\{1, 2, \dots, n\}$ .

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223 ► **Definition 3.5** (Shallow Orders). *Let  $f: X \rightarrow \mathbb{R}$  be a filter on a Lefschetz complex, and*  
224  *$n = \#\text{BD}(f)$  its number of birth-death pairs. A shallow order is a bijection  $\Phi: [n] \rightarrow \text{BD}(f)$*   
225 *such that  $\varphi_i = \Phi(i)$  is shallow after canceling  $\varphi_1$  to  $\varphi_{i-1}$ , for each  $1 \leq i \leq n$ .*

226 Two particular shallow orders will be instrumental in the study of the dependencies between  
227 birth-death points, resp. their cancellations. To introduce them, we write  $\varphi_i^\circ$  and  $\varphi_i^\times$  for  
228 the birth-giving and death-giving cells of a birth-death pair  $\varphi_i$ . The first such special order  
229 prefers late births over early births, while the second prefers early deaths over late deaths:

$$230 \quad A: [n] \rightarrow \text{BD}(f) \quad \text{such that} \quad f(\alpha_i^\circ) > f(\alpha_{i+1}^\circ) \quad \text{for } 1 \leq i < n; \quad (3)$$

$$231 \quad \Omega: [n] \rightarrow \text{BD}(f) \quad \text{such that} \quad f(\omega_i^\times) < f(\omega_{i+1}^\times) \quad \text{for } 1 \leq i < n, \quad (4)$$

232 in which  $A$  and  $\Omega$  are bijections, and we write  $\alpha_i = A(i)$  and  $\omega_i = \Omega(i)$ . Note that  $\alpha_1^\circ$  is the  
233 last birth-giving cell in the filter, and  $\alpha_1^\times$  is its first coface, which implies that  $\alpha_1$  is shallow.  
234 After canceling  $\alpha_1$ ,  $\alpha_2$  is shallow, etc., so  $A$  is indeed a shallow order. Symmetrically,  $\omega_1^\times$  is  
235 the first death-giving cell, and  $\omega_1^\circ$  is its last facet, which again implies that  $\omega_1$  is shallow. By  
236 canceling  $\omega_1$  and iterating, we conclude that  $\Omega$  is also a shallow order.

237 *Example.* We continue the example from Section 3.1 considering the 16 minima and maxima  
238 of the function on the circle shown in Figure 1. They form 7 birth-death pairs, with one  
239 minimum and one maximum unpaired. Following the two special shallow orders, we get

$$240 \quad A([7]) = ((\mathbf{h}, \mathbf{G}), (\mathbf{e}, \mathbf{E}), (\mathbf{d}, \mathbf{C}), (\mathbf{f}, \mathbf{F}), (\mathbf{b}, \mathbf{B}), (\mathbf{c}, \mathbf{D}), (\mathbf{g}, \mathbf{H})); \quad (5)$$

$$241 \quad \Omega([7]) = ((\mathbf{b}, \mathbf{B}), (\mathbf{d}, \mathbf{C}), (\mathbf{f}, \mathbf{F}), (\mathbf{e}, \mathbf{E}), (\mathbf{c}, \mathbf{D}), (\mathbf{h}, \mathbf{G}), (\mathbf{g}, \mathbf{H})), \quad (6)$$

242 in which we write the elements of the images in sequence from 1 to 7. Note that both orders  
243 are linear extensions of the poset displayed in the lower right panel of Figure 1, a property  
244 we will explore next.

## 245 4 The Depth Poset

246 This section introduces the main new concept of this paper, the depth poset of a filter,  
247 which is a formalization of the dependencies between the birth-death pairs, respectively  
248 their cancellations. After defining the poset and proving some of its pertinent properties, we  
249 explain how to construct it, and finally establish the correctness of the algorithm.

### 250 4.1 Partial Order on Birth-death Pairs

251 A shallow order is a total order on the birth-death pairs or, equivalently, a complete graph  
252 with vertices  $\text{BD}(f)$  whose edges are directed in an acyclic manner. The intersection of two  
253 such graphs corresponds to a partial order such that both total orders are linear extensions  
254 of the poset. We apply this construction to the set of all shallow orders of a filter.

255 ► **Definition 4.1** (Depth Poset). *Letting  $f: X \rightarrow \mathbb{R}$  be a filter on a Lefschetz complex, the*  
256 *depth poset, denoted  $\text{Depth}(f)$ , is the intersection of all shallow orders on  $\text{BD}(f)$ .*

257 Its full name would be the *depth poset of canceling shallow birth-death pairs in a Lefschetz*  
258 *complex*. By definition, it is the largest partial order on  $\text{BD}(f)$  such that every shallow order  
259 is a linear extension. Note that  $f(\varphi^\circ) > f(\psi^\circ)$  if  $\varphi$  precedes  $\psi$  in  $A$ , and  $f(\varphi^\times) < f(\psi^\times)$  if  
260  $\varphi$  precedes  $\psi$  in  $\Omega$ . Since  $A$  and  $\Omega$  are particular shallow orders,  $(\varphi, \psi) \in \text{Depth}(f)$  implies  
261 that  $\varphi$  precedes  $\psi$  in both, so the pairs are necessarily nested:

262 ► **Proposition 4.2.** *Let  $f: X \rightarrow \mathbb{R}$  be a filter on a Lefschetz complex, and  $\varphi, \psi$  two pairs in*  
 263  *$\text{BD}(f)$ . Then  $f(\psi^\circ) < f(\varphi^\circ) < f(\varphi^\times) < f(\psi^\times)$  whenever  $(\varphi, \psi) \in \text{Depth}(f)$ .*

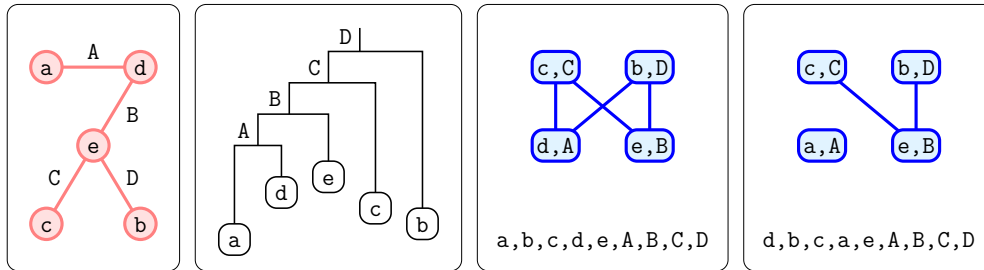
264 Observe that this implies that an ordering of the birth-death pairs by the difference between  
 265 birth and death is necessarily a linear extension of the poset. We thus introduce

$$266 \quad \Pi: [n] \rightarrow \text{BD}(f) \text{ such that } f(\pi_i^\times) - f(\pi_i^\circ) \leq f(\pi_{i+1}^\times) - f(\pi_{i+1}^\circ) \text{ for } 1 \leq i < n, \quad (7)$$

267 in which  $\Pi$  is a bijection and we write  $\pi_i = \Pi(i)$ . Note that  $\Pi$  orders the birth-death pairs  
 268 by persistence. Proposition 4.2 thus implies that the pairs can be canceled in this sequence  
 269 without otherwise affecting the filter:

270 ► **Corollary 4.3.** *Let  $f: X \rightarrow \mathbb{R}$  be a filter on a Lefschetz complex, and  $\Pi$  the ordering of the*  
 271 *birth-death pairs by persistence. Then  $\Pi$  is a shallow order of  $f$ .*

272 *Example.* We note that the depth poset of the function on the circle in Figure 1 is consistent  
 273 with the merge tree of the function; see e.g. [20]. There are however filtered graphs (1-  
 274 dimensional Lefschetz complexes) with isomorphic merge trees for which the depth posets  
 275 display different dependencies; see Figure 5. There are also examples of filtered graphs that  
 have different merge trees but isomorphic depth posets (not shown).



276 ■ Figure 5: *From left to right: a filtered graph, its merge tree (dendrogram), its depth poset above the nodes and arcs ordered according to the filter, and the depth poset after swapping nodes  $a$  and  $d$  in the filter. The swap only causes nodes  $a$  and  $d$  to trade places in the dendrogram, which does not affect the structure of the merge tree. In contrast, it shrinks the depth poset from four to two relations.*

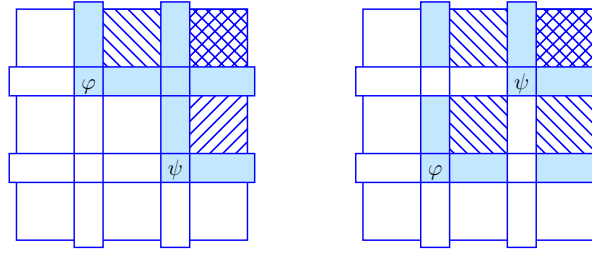
## 277 4.2 Transposing and Canceling Shallow Pairs

278 The conclusion that all linear extensions of the depth poset are shallow orders is not immediate  
 279 since the definition of this poset is indirect. We therefore go slow and first establish some  
 280 basic properties. Let  $\Phi: [n] \rightarrow \text{BD}(f)$  be a total order on the birth-death pairs of a filter. A  
 281 *transposition* at positions  $1 \leq k, k+1 \leq n$  produces another total order in which  $\Phi(k)$  and  
 282  $\Phi(k+1)$  are swapped. We are primarily interested in transpositions that swap shallow pairs.

283 ► **Lemma 4.4.** *Let  $f: X \rightarrow \mathbb{R}$  be a filter on a Lefschetz complex, and  $\Phi, \Psi: [n] \rightarrow \text{BD}(f)$*   
 284 *total orders that differ by the transposition at positions  $1 \leq k, k+1 \leq n$ . If  $\Phi$  is a shallow*  
 285 *order, and after canceling the first  $k-1$  pairs,  $\Phi(k+1)$  is a shallow pair, then  $\Psi$  is also a*  
 286 *shallow order, and the quotients after canceling the first  $i$  pairs of  $\Phi$  and  $\Psi$ , respectively, are*  
 287 *the same for all  $i \neq k$ .*

288 **Proof.** The claim about the quotients is trivially true for  $i < k$ . For the next step, assume  
 289  $i = k+1$  and write  $\varphi = \Phi(k)$  and  $\psi = \Phi(k+1)$ . After canceling the first  $k-1$  pairs,  $\varphi$  is  
 290 shallow because  $\Phi$  is a shallow order, and  $\psi$  is shallow by assumption. We transpose the

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■ Figure 6: Two shallow pairs with possibly non-zero entries in the *shaded* portions of their rows and columns. On the *left*, the two pairs have either disjoint or incomparable intervals:  $f(\varphi^\circ) < f(\psi^\circ)$  and  $f(\varphi^\times) < f(\psi^\times)$ , and on the *right*, their intervals are nested:  $f(\psi^\circ) < f(\varphi^\circ) < f(\varphi^\times) < f(\psi^\times)$ .

291 two pairs and argue that the swap does not affect the boundary matrix or, equivalently, the  
 292 quotient after the  $k + 1$  cancellations. To see this, we write  $\text{pvt}(\varphi)$  for the entry common to  
 293 row  $\varphi^\circ$  and column  $\varphi^\times$ , and note that it suffices to look at the entries to the upper right  
 294 of  $\text{pvt}(\varphi)$  and  $\text{pvt}(\psi)$ : the cross-hatched regions in Figure 6. Let  $t$  be the column of such  
 295 an entry. Setting  $\ell = \Delta_{k-1}[\varphi^\circ, t] + \Delta_{k-1}[\varphi^\circ, \psi^\times]$  and  $m = \Delta_{k-1}[\psi^\circ, t] + \Delta_{k-1}[\psi^\circ, \varphi^\times]$ , the  
 296 effect of the two cancellations is adding  $\ell$  times column  $\varphi^\times$  and  $m$  times column  $\psi^\times$  to  
 297 column  $t$ . This is clear if the respective second terms are zero. The only other case is when  
 298  $\Delta_{k-1}[\varphi^\circ, \psi^\times] = 1$ , which may happen in the configuration illustrated on the left in Figure 6.  
 299 If we first cancel  $\psi$ , then this changes the parity of  $\Delta_{k-1}[\varphi^\circ, t]$ , and if we first cancel  $\varphi$ ,  
 300 then we add column  $\varphi^\times$  to column  $\psi^\times$  before possibly adding it to column  $t$ . Either way,  
 301 the effect is the same. Since  $\ell$  and  $m$  are independent of the order in which  $\varphi$  and  $\psi$  are  
 302 canceled, the quotients after canceling  $i = k + 1$  pairs in  $\Phi$  and  $\Psi$  agree. For trivial reasons,  
 303 the quotients therefore also agree after canceling  $i > k + 1$  pairs each, which implies that  $\Psi$   
 304 is also a shallow order, as claimed. ◀

305 Given a finite poset, it is not difficult to impose an acyclic relation on its linear extensions,  
 306 such that two related extensions differ by a single transposition, and there is only a single  
 307 maximum. We prove a similar result for the shallow orders, where we face the difficulty that  
 308 we do not yet have a poset whose linear extensions are exactly the shallow orders.

309 ▶ **Lemma 4.5.** *Let  $f: X \rightarrow \mathbb{R}$  be a filter on a Lefschetz complex, and  $\Phi, \Psi: [n] \rightarrow \text{BD}(f)$   
 310 two shallow orders. Then there is a sequence of shallow orders,  $\Phi = \Phi_0, \Phi_1, \dots, \Phi_m = \Psi$   
 311 such that  $\Phi_{k-1}, \Phi_k$  differ by a single transposition, for any  $1 \leq k \leq m$ .*

312 **Proof.** We fix a shallow order, which we construct iteratively by canceling all shallow pairs.  
 313 Indeed, which birth-death pairs are shallow depends solely on  $f_0 = f$ . After canceling these  
 314 shallow pairs, we get a filter  $f_1: X_1 \rightarrow \mathbb{R}$  which, by Lemma 4.4, does not depend on the  
 315 sequence in which we cancel these pairs. Next, we cancel the shallow pairs of  $f_1$  to get  
 316  $f_2: X_2 \rightarrow \mathbb{R}$ , etc. Let  $\Xi: [n] \rightarrow \text{BD}(f)$  be the sequence in which the pairs are canceled, and  
 317 write  $\xi_i = \Xi(i)$  for the  $i$ -th pair. By construction,  $\Xi$  is a shallow order of  $f$ .

318 Writing  $\varphi_i = \Phi(i)$ , we construct a sequence of transpositions that transform  $\Phi$  into  $\Xi$ . In  
 319 each iteration, we let  $\xi_i$  be the first pair in which  $\Xi$  differs from  $\Phi$ , set  $\varphi_k = \xi_i$ , and move  $\varphi_k$   
 320 forward into  $i$ -th position using transpositions. By Lemma 4.4, each transposition produces  
 321 a new shallow order, provided the two pairs are shallow prior to their transposition and  
 322 after canceling all preceding pairs. But this is clear because the second of the two pairs is  
 323  $\xi_i$ , which is shallow after canceling the first  $i - 1$  pairs in  $\Xi$ . By construction, these  $i - 1$   
 324 pairs are also the first  $i - 1$  predecessors in  $\Phi$ . We thus get a sequence of transpositions that  
 325 transform  $\Phi$  into  $\Xi$ , while each step preserves the property of the linear extension being

326 shallow. Similarly, we construct such a sequence for  $\Psi$  and  $\Xi$ , and append its reverse to get  
 327 a sequence of transpositions that transforms  $\Phi$  into  $\Psi$ , as required. ◀

328 Call a terminal sequence of pairs in a shallow order a *suffix*, and the initial remainder the  
 329 *complementary prefix*. We use the last two lemmas to show that the quotient after canceling  
 330 all pairs in the prefix does not depend on the order in which the pairs are canceled.

331 ▶ **Lemma 4.6.** *Let  $f: X \rightarrow \mathbb{R}$  be a filter on a Lefschetz complex, and  $\Phi, \Psi: [n] \rightarrow \text{BD}(f)$   
 332 two shallow orders that share a common suffix. Then  $X' = X''$ , in which  $f': X' \rightarrow \mathbb{R}$  and  
 333  $f'': X'' \rightarrow \mathbb{R}$  are the filters on the quotients after canceling the pairs in the complementary  
 334 prefix of  $\Phi$  and  $\Psi$ , respectively. Furthermore, the common depth poset of  $f'$  and  $f''$  is the  
 335 depth poset of  $f$  restricted to  $\text{BD}(f') = \text{BD}(f'')$ .*

336 **Proof.** By Lemma 4.5, we can transform the prefix of  $\Phi$  into the prefix of  $\Psi$  by a sequence  
 337 of transpositions that leaves the common suffix untouched. By Lemma 4.4, each such  
 338 transposition preserves the quotients after canceling the transposed pair and all their  
 339 predecessors. This implies that the quotient after canceling all pairs in the prefix remains  
 340 constant throughout the sequence of transpositions. The claim about the depth poset follows  
 341 because the shallow orders of  $f' = f''$  are exactly the suffixes of the shallow orders of  $f$ . ◀

342 It will also be useful to have the following claim about the preservation of the row of the  
 343 last birth-giving cell and the column of the first death-giving cell.

344 ▶ **Lemma 4.7.** *Let  $f: X \rightarrow \mathbb{R}$  be a filter on a Lefschetz complex,  $\alpha_1 = A^{-1}(1)$  and  
 345  $\omega_1 = \Omega^{-1}(1)$  the respective leading pairs of the two special shallow orders,  $\psi \neq \alpha_1, \omega_1$  a  
 346 birth-death pair, and  $\gamma \neq \psi$  a shallow such pair. Then the cancellation of  $\gamma$  preserves  
 347  $\Delta[\alpha_1^\circ, \psi^\times]$  and  $\Delta[\psi^\circ, \omega_1^\times]$ , with  $\gamma \neq \alpha_1$  in the first case and  $\gamma \neq \omega_1$  in the second case.*

348 **Proof.** The two claims are symmetric, so it suffices to prove the first. To have an effect  
 349 on the entries in row  $\alpha_1^\circ$ , it is necessary that  $\Delta[\alpha_1^\circ, \gamma^\times] = 1$ . But since  $\alpha_1$  and  $\gamma$  are both  
 350 shallow, this is only possible if  $f(\alpha_1^\circ) < f(\gamma^\circ)$ , which contradicts  $\alpha_1 = A^{-1}(1)$ . ◀

### 351 4.3 Lazy Reduction with Clearing

370 To construct the depth poset, we use variants of the standard matrix reduction algorithm  
 371 for persistent homology; see e.g. [8, Chapter VII]. We begin with the variant of the column  
 372 reduction algorithm, which differs from the classic algorithm in two ways. From [7] it borrows  
 373 the idea that the columns can be reduced by taking the leftmost non-zero entries (pivots) in  
 374 the rows from bottom to top. These pivots correspond to the birth-death pairs and their  
 375 ordering prefers late over early births; compare with the first special shallow order,  $A$ , defined  
 376 in (3). Note, however, that the birth-death pairs do not have to be known ahead of time  
 377 as they are implicitly detected by the algorithm. From [3] it borrows the idea that rows  
 378 and columns can be cleared after the corresponding pivot has been established. Indeed,  
 379 after canceling two cells, we delete the corresponding rows and columns to get the boundary  
 380 matrix of the quotient. At the same time, we collect the relations in an initially empty list;  
 381 see Algorithm 1. By choice of the pivot, each pair  $(s, t)$  is shallow when it is visited, and the  
 382 ensuing column operations effectively cancel  $s$  and  $t$ ; see Definition 2.3 and Figure 2. By  
 383 prioritizing late births, Algorithm 1 visits the pairs according to the first special shallow  
 384 order and thus computes  $A$ . Letting  $\Delta'_i$  be the matrix  $\Delta'$  after  $i$  iterations of Algorithm 1,  
 385 this is therefore the boundary matrix of the quotient after canceling  $\alpha_1$  through  $\alpha_i$ . There  
 386 are shallow pairs as long as there are birth-death pairs, so  $\Delta'_i \neq 0$ , for all  $i < n$ , and  $\Delta'_n = 0$ .  
 387 Hence, Algorithm 1 halts after  $n = \#\text{BD}(f)$  iterations.

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### 352 ■ Algorithm 1 Bottom to Top Column Reduction

---

```

353 1:  $\Delta' = \Delta; B' = \emptyset; i = 0;$ 
354 2: while  $\Delta' \neq 0$  do  $i = i + 1;$ 
355 3:   let  $\Delta'[s, t]$  be leftmost non-zero entry in last non-zero row;  $\alpha_i = (s, t);$ 
356 4:   while  $\exists y > t$  such that  $\Delta'[s, y] = 1$  do
357 5:     add column  $t$  to column  $y$  in  $\Delta'$ ; append  $(t, y)$  to  $B'$ 
358 6:   end while;
359 7:   delete rows  $s$  and  $t$  and columns  $s$  and  $t$  from  $\Delta'$ 
360 8: end while.

```

---

### 361 ■ Algorithm 2 Left to Right Row Reduction

---

```

362 1:  $\Delta'' = \Delta; B'' = \emptyset; j = 0;$ 
363 2: while  $\Delta'' \neq 0$  do  $j = j + 1;$ 
364 3:    $\Delta''[s, t]$  is lowest non-zero entry in first non-zero column;  $\omega_j = (s, t);$ 
365 4:   while  $\exists x < s$  such that  $\Delta''[x, t] = 1$  do
366 5:     add row  $s$  to row  $x$  in  $\Delta''$ ; append  $(s, x)$  to  $B''$ ;
367 6:   end while;
368 7:   delete rows  $s$  and  $t$  and columns  $s$  and  $t$  from  $\Delta''$ 
369 8: end while.

```

---

388 Symmetrically, Algorithm 2 computes  $\Omega$  while reducing the boundary matrix with row  
389 operations. Letting  $\Delta''_j$  be the matrix  $\Delta''$  after  $j$  iterations of Algorithm 2, it is the boundary  
390 matrix of the quotient after canceling  $\omega_1$  through  $\omega_j$ , and Algorithm 2 also halts after  $n$   
391 iterations. We state this for later reference:

392 **► Lemma 4.8.** *Let  $f: X \rightarrow \mathbb{R}$  be a filter on a Lefschetz complex, and  $\Delta'_i, \Delta''_j$  as defined*  
393 *above. Then  $\Delta'_i$  is the boundary matrix of the quotient after canceling  $\alpha_1, \alpha_2, \dots, \alpha_i$ , and  $\Delta''_j$*   
394 *is the boundary matrix of the quotient after canceling  $\omega_1, \omega_2, \dots, \omega_j$ .*

## 395 4.4 Relations from Book-keeping

396 The relations collected by the two algorithms do not contradict each other: if  $(\varphi^\times, \psi^\times)$  is in  
397 the transitive closure of  $B'$ , then  $f(\psi^\circ) < f(\varphi^\circ) < f(\varphi^\times) < f(\psi^\times)$ , and if  $(\psi^\circ, \varphi^\circ)$  is in the  
398 transitive closure of  $B''$ , then we get the reverse of these inequalities. This excludes their  
399 co-occurrence, so it makes sense to define the transitive closure of the union:

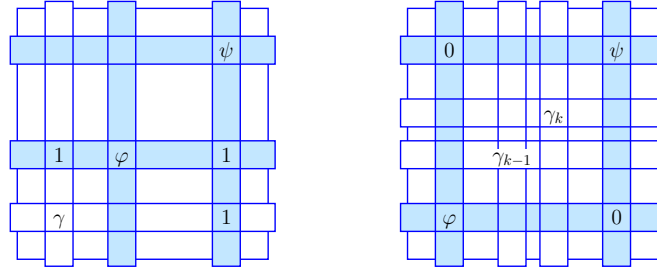
$$400 \quad R(f) = \text{closure}\{(\varphi, \psi) \mid (\varphi^\times, \psi^\times) \in B' \text{ or } (\varphi^\circ, \psi^\circ) \in B''\}. \quad (8)$$

401 We claim that  $R(f)$  is in fact the depth poset of  $f$ . While this is plausible, it is not obvious  
402 and requires a proof.

403 **► Theorem 4.9.** *Let  $f: X \rightarrow \mathbb{R}$  be a filter on a Lefschetz complex. Then  $\text{Depth}(f) = R(f)$ .*

404 **Proof.** We first prove  $R(f) \subseteq \text{Depth}(f)$ . It suffices to argue the containment for the relations  
405 supplied by  $B'$ , with the argument for  $B''$  being symmetric. Let therefore  $(\varphi, \psi) \in R(f)$   
406 and recall that  $(\varphi^\circ, \psi^\times) \in B'$  iff  $\Delta'_i[\varphi^\circ, \psi^\times] = 1$ ; see Lines 4 and 5 of Algorithm 1. Set  
407  $i + 1 = A^{-1}(\varphi)$  and write  $f'_i: X'_i \rightarrow \mathbb{R}$  for the filter after canceling  $\alpha_1, \alpha_2, \dots, \alpha_i$ . By  
408 Lemma 4.6,  $(\varphi, \psi) \in \text{Depth}(f)$  iff  $(\varphi, \psi) \in \text{Depth}(f'_i)$ , so we can focus on the situation after  
409 canceling the first  $i$  pairs in  $A$ . The pivots are processed from bottom to top, which implies  
410  $f(\varphi^\circ) > f(\psi^\circ)$ ; that is:  $\text{pvt}(\varphi)$  is below  $\text{pvt}(\psi)$ , as in Figure 7 on the left. The question is

411 whether there is any sequence of pairs of  $f'_i$ —excluding  $\varphi$ —whose cancellation makes  $\psi$  a  
 412 shallow pair, so it can be canceled before  $\varphi$ . For such a sequence to exist, there must be a  
 413 pair,  $\gamma$ , whose cancellation changes  $\Delta'[\varphi^\circ, \psi^\times]$  from 1 to 0. However, since  $\varphi$  is the leading  
 pair in the first special shallow order of  $f'_i$ , this is prohibited by Lemma 4.7.



■ Figure 7: The matrix on the *left* illustrates first part of the proof: to cancel  $\gamma$ , we would add its column to the column of  $\psi$ , with the effect that  $\Delta'[\varphi^\circ, \psi^\times]$  changes from 1 to 0. But such  $\gamma$  does not exist after canceling all pairs below  $\text{pvt}(\varphi)$ . The matrix on the *right* illustrates the second part of the proof: after transposing  $\gamma_{k-1}$  and  $\gamma_k$ , we are one step closer to a contradiction.

414

415 We second prove  $\text{Depth}(f) \subseteq R(f)$ . It suffices to argue the containment for pairs  
 416  $(\varphi, \psi) \in \text{Depth}(f)$  that are not implied by transitivity. Set  $i+1 = A^{-1}(\varphi)$  and  $j+1 = \Omega^{-1}(\varphi)$ ,  
 417 write  $f'_i: X'_i \rightarrow \mathbb{R}$  and  $\Delta'_i$  after canceling the prefix of length  $i$  in  $A$ ,  $f''_j: X''_j \rightarrow \mathbb{R}$  and  $\Delta''_j$   
 418 after canceling the prefix of length  $j$  in  $\Omega$ , and  $f_{ij}: X_{ij} \rightarrow \mathbb{R}$  and  $\Delta_{ij}$  after canceling the  
 419 pairs in both prefixes. By Lemmas 4.6 and 4.7, we have  $\Delta_{ij}[\varphi^\circ, \psi^\times] = \Delta'_i[\varphi^\circ, \psi^\times]$  and  
 420  $\Delta_{ij}[\psi^\circ, \varphi^\times] = \Delta''_j[\psi^\circ, \varphi^\times]$ . If either of these two entries is 1, then  $(\varphi, \psi) \in R(f)$  and we  
 421 are done. Hence, assume  $\Delta_{ij}[\varphi^\circ, \psi^\times] = \Delta_{ij}[\psi^\circ, \varphi^\times] = 0$ , as in Figure 7 on the right.  
 422 Set  $n_{ij} = \#\text{BD}(f_{ij})$ , and let  $\Phi: [n_{ij}] \rightarrow \text{BD}(f_{ij})$  be a linear extension of  $\text{Depth}(f_{ij})$  that  
 423 minimizes  $m = \Phi^{-1}(\psi) - \Phi^{-1}(\varphi)$ . We already established  $R(f_{ij}) \subseteq \text{Depth}(f_{ij})$ , so  $\Phi$  is  
 424 also a linear extension of  $R(f_{ij})$ . If  $m = 1$ , then we can transpose  $\varphi$  and  $\psi$  and thus  
 425 contradict  $(\varphi, \psi) \in \text{Depth}(f_{ij})$ . So assume  $m \geq 2$  and write  $\varphi = \gamma_0, \gamma_1, \dots, \gamma_m = \psi$  for  
 426 the relevant subsequence. Since  $(\varphi, \psi)$  is not implied by transitivity, there is no chain of  
 427 two or more relations that connects  $\varphi$  to  $\psi$  in the depth poset. Hence, there is an index  
 428  $1 \leq k \leq m$  such that  $(\gamma_{k-1}, \gamma_k) \notin \text{Depth}(f_{ij})$ , and since  $R(f_{ij}) \subseteq \text{Depth}(f_{ij})$ , we also have  
 429  $(\gamma_{k-1}, \gamma_k) \notin R(f_{ij})$ . If  $(\varphi, \gamma_{k-1})$  or  $(\gamma_k, \psi)$  is a relation in  $\text{Depth}(f_{ij})$ , then we transpose  
 430  $\gamma_{k-1}$  and  $\gamma_k$  and get a new linear extension of  $\text{Depth}(f_{ij})$  and  $R(f_{ij})$ . There is necessarily at  
 431 least one pair to transpose, so if we iterate, the predecessors of  $\psi$  migrate to the left, and the  
 432 successors of  $\varphi$  migrate to the right. Eventually, we get a transposition that involves  $\varphi = \gamma_0$   
 433 or  $\psi = \gamma_m$ , and either way, we get a linear extension that contradicts our choice of  $\Phi$ . ◀

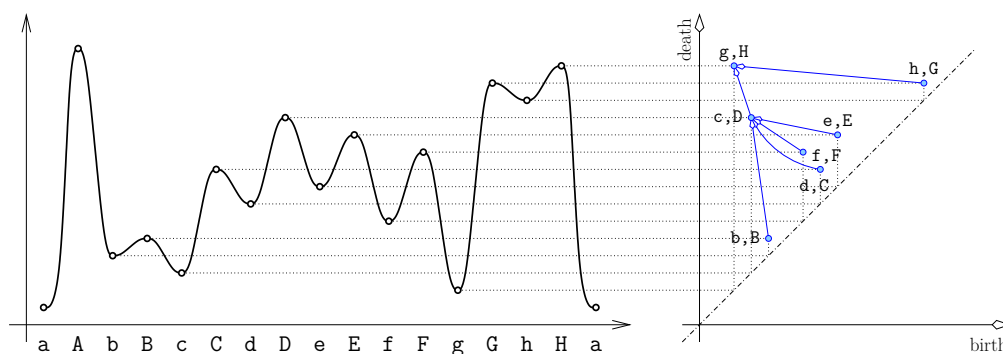
434 By definition of the depth poset, every shallow order of  $f$  is a linear extension of  $\text{Depth}(f)$ .  
 435 Using Theorem 4.9, it is not difficult to argue also the converse, which implies that the  
 436 shallow orders and the linear extensions of the depth poset are indeed one and the same.

437 Observe that Algorithm 1 adds a column  $t$  to another column  $y$  only if the two cells satisfy  
 438  $\dim t = \dim y$ . Similarly, Algorithm 2 adds a row  $s$  to another row  $x$  only if  $\dim s = \dim x$ .  
 439 By Theorem 4.9, this implies that every relation in  $\text{Depth}(f)$  is between birth-death pairs  
 440 of the same pair of dimensions. Hence, the depth poset is a disjoint union of one poset per  
 441 pair of consecutive dimensions. This motivates us to define  $\text{BD}_p(f) \subseteq \text{BD}(f)$  as the birth  
 442 death-pairs  $\varphi = (\varphi^\circ, \varphi^\times)$  with  $\dim \varphi^\circ = \dim \varphi^\times - 1 = p$ , and similarly restrict the depth  
 443 poset by defining  $\text{Depth}_p(f) = \text{Depth}(f) \cap (\text{BD}_p(f) \times \text{BD}_p(f))$ .

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444 ► **Corollary 4.10.** *Let  $f: X \rightarrow \mathbb{R}$  be a filter on a Lefschetz complex of dimension  $d$ . Then*  
445  $\text{Depth}(f) = \text{Depth}_0(f) \sqcup \text{Depth}_1(f) \sqcup \dots \sqcup \text{Depth}_{d-1}(f)$ .

446 This corollary is relevant if we annotate a persistence diagram with arcs that connect points  
447 representing birth-death pairs related to each other in the depth poset; see Figure 8 for the  
448 0-dimensional persistence diagram of a 1-dimensional function. By Corollary 4.10, each such  
arc belongs to a unique dimension.



■ Figure 8: The function on the circle introduced in Figure 1 on the *left*, and its persistence diagram overlaid with the depth poset on its birth-death pairs on the *right*.

449

## 5 Discussion

450

451 The main contribution of this paper are the introduction of the depth poset—which records  
452 and organizes the dependencies between the cancellations of shallow birth-death pairs in a  
453 Lefschetz complex—and a proof that it can be constructed by a customized but otherwise  
454 straightforward matrix reduction algorithm. The novel structure raises a number of questions  
455 and opens opportunities for further work:

- 456 ■ It would be interesting to perform stochastic experiments to understand the statistical  
457 behavior of the depth poset. Are differences in the local structure of the relations helpful  
458 in detecting outliers or in the reduction of noise in sampled data?
- 459 ■ It would also be interesting to analyze the sensitivity of the depth poset to transpositions  
460 in the filter, as this may be correlated with the changing dynamics of the gradient flow;  
461 see [6, 15, 17] for related work in this direction.

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462

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