

From Hitchin systems to quantum groups

Joint works with Jakub Löwit and Shon Ngo

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STABLE BUNDLES AND INTEGRABLE SYSTEMS

NIGEL HITCHIN

§1. Introduction. The moduli spaces of stable vector bundles over a Riemann surface are algebraic varieties of a very special nature. They have been studied for the past twenty years from the point of view of algebraic geometry, number theory and the Yang-Mills equations. We adopt here another viewpoint, considering the *symplectic geometry* of their cotangent bundles. These turn out to be algebraically completely integrable Hamiltonian systems in a very natural way. For rank 2 bundles of odd degree this appeared as a byproduct of an investigation [5] into certain solutions of the self-dual Yang-Mills equations, a subject on which Yuri Manin has had a profound influence.

The cotangent bundle T^*N of an n -dimensional complex manifold N is a completely integrable Hamiltonian system if there exist n functionally independent, Poisson-commuting holomorphic functions on T^*N . These functions for the case where N is the moduli space of stable G -bundles on a compact Riemann surface M , and G a complex semisimple Lie group are easy to describe. The tangent space of the moduli space at a point is identified with the sheaf cohomology group $H^1(M; \mathfrak{g})$ where $\mathfrak{g}$ is a holomorphic bundle of Lie algebras. By Serre duality the cotangent space is $H^0(M; \mathfrak{g} \otimes K)$. An invariant polynomial of degree d on the Lie algebra then gives rise to a map from this cotangent space to the space $H^0(M; K^d)$ of differentials of degree d on M . Taking a basis for the ring of invariant polynomials yields a map to the vector space $W = \bigoplus_{d=1}^r H^0(M; K^d)$ where d_i are the degrees of the basic invariant polynomials. Somewhat miraculously, the dimension of this vector space is always equal to the dimension of the moduli space N , thus providing the n functions.

The Hamiltonian vector fields corresponding to these functions give n commuting vector fields along the fibres of the map to W . The system is called *algebraically completely integrable* if the generic fibre is an open set in an abelian variety and the vector fields are linear. For the system above, at least for the case where G is a classical group, this also turns out to be true, the abelian variety being either a Jacobian or a Prym variety of a curve covering M . The construction of this curve, and its corresponding Jacobian, parallels the solution of differential equations of "spinning top" type involving isospectral deformations of a matrix of polynomials in one variable. A point of the cotangent bundle of the moduli space consists of a stable vector bundle V (with G -structure) and a holomorphic section $\Phi \in H^0(M; \mathfrak{g} \otimes K)$, which gives a holomorphic map $\Phi: V \rightarrow V \otimes K$. We form the curve of eigenvalues S defined by the equation

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De Concini–Procesi positive Borel for $U_q(\mathrm{GL}_n)$

- (Drinfeld/Jimbo, 1985) $U_{\mathbb{F}}^{\geq 0}$, $\mathbb{F} := \mathbb{Q}(q)$: generated by K_{λ}^{+} , $\lambda \in \mathbb{Z}^n$, and by E_i , $1 \leq i < n$, subject to

$$K_{\lambda}^{+} K_{\mu}^{+} = K_{\lambda+\mu}^{+}, \quad K_{\lambda}^{+} E_i K_{-\lambda}^{+} = q^{(\lambda, \alpha_i)} E_i, \quad E_i E_j = E_j E_i \quad (|i - j| > 1),$$

$$E_i^2 E_j - (q + q^{-1}) E_i E_j E_i + E_j E_i^2 = 0 \quad (|i - j| = 1).$$

- with Hopf algebra structure, $\Delta : U_{\mathbb{F}}^{\geq 0} \rightarrow U_{\mathbb{F}}^{\geq 0} \otimes U_{\mathbb{F}}^{\geq 0}$ coproduct, $\epsilon : U_{\mathbb{F}}^{\geq 0} \rightarrow \mathbb{F}$ counit and $S : U_{\mathbb{F}}^{\geq 0} \rightarrow U_{\mathbb{F}}^{\geq 0}$ antipode:

$$\Delta(K_{\lambda}^{+}) = K_{\lambda}^{+} \otimes K_{\lambda}^{+}, \quad \Delta(E_i) = 1 \otimes E_i + E_i \otimes K_{\alpha_i}^{+},$$

$$\epsilon(K_{\lambda}^{+}) = 1, \quad \epsilon(E_i) = 0, \quad S(K_{\lambda}^{+}) = K_{-\lambda}^{+}, \quad S(E_i) = -E_i K_{-\alpha_i}^{+}.$$

- De Concini–Procesi (1993) integral form: $U_{\mathbb{A}}^{\geq 0} \subset U_{\mathbb{F}}^{\geq 0}$ the $\mathbb{A} := \mathbb{Z}[q^{\pm 1}]$ -subalgebra generated by the K_{λ}^{+} and by $e_{ab} := (q - q^{-1}) E_{ab}$, $1 \leq a < b \leq n$, where $E_{i,i+1} := E_i$ and, for $b > a + 1$, $E_{ab} := E_{a,b-1} E_{b-1,b} - q^{-1} E_{b-1,b} E_{a,b-1}$.

The extended Lusztig negative Borel

- V finite free $\mathbb{A}$ -module with $U_{\mathbb{A}}^{\geq 0}$ -action, type 1:

$$V_{\mathbb{F}} := V \otimes_{\mathbb{A}} \mathbb{F} = \bigoplus_{\lambda \in \mathbb{Z}^n} (V_{\mathbb{F}})_{K_a^+ = q^{\lambda} a}.$$

- For $m \in V$, $\ell \in V^* = \text{Hom}_{\mathbb{A}}(V, \mathbb{A})$, put $c_{\ell, m}(x) = \ell(xm)$,

$$U_{\mathbb{A}}^{\leq 0} := \text{span}_{\mathbb{A}}\{c_{\ell, m}\} \subseteq \text{Hom}_{\mathbb{A}}(U_{\mathbb{A}}^{\geq 0}, \mathbb{A}).$$

Its coefficient Hopf structure and pairing $\tau(x, \phi) = \phi(x)$ are

$$(\phi\psi)(x) = (\phi \otimes \psi)(\Delta x), \quad (\Delta\phi)(x \otimes y) = \phi(yx).$$

- $U_{\mathbb{A}}^{\leq 0}$ generated by modified divided powers (Lusztig, 1993):

$$F_i^{(r)} := \frac{F_i^r}{\prod_{s=1}^r \frac{1-q^{-2s}}{1-q^{-2}}}, \quad (K_a^-)^{\pm 1}, \quad \prod_{j=0}^{r-1} \frac{K_a^- - q^j}{q^r - q^j},$$

where $F_i := \sum_{\lambda \in \mathbb{Z}^n} (K_{\lambda}^+ e_i)^*$ and $1 \leq i < n$, $1 \leq a \leq n$, $r \geq 1$.

Hybrid family quantum group is the Drinfeld double

- *Hopf pairing* $\tau : U_{\mathbb{A}}^{\geq 0} \times U_{\mathbb{A}}^{\leq 0} \rightarrow \mathbb{A}$, $\tau(x, \phi) = \phi(x)$:
 $\tau(K_{\lambda}^{+}, K_{\mu}^{-}) = q^{-\langle \lambda, \mu \rangle}$, $\tau(e_i, F_j) = \delta_{ij}$, $\tau(e_i, K_{\mu}^{-}) = 0 = \tau(K_{\lambda}^{+}, F_j)$.

- *Drinfeld double* $\mathbb{U} := U_{\mathbb{A}}^{\geq 0} \otimes_{\mathbb{A}} U_{\mathbb{A}}^{\leq 0}$, $\mathbb{A}$ -Hopf algebra:

$$[e_i, F_j] = \delta_{ij}(K_{\alpha_i}^{+} - K_{\alpha_i}^{-}), \quad [K_{\lambda}^{+}, K_{\mu}^{-}] = 0,$$

hybrid family quantum group (Hausel–Löwitt, 2026).

- $t_a := K_a^{+} K_a^{-}$ central group-like,
 $\mathbb{A}_t := \mathbb{A}[t_a, t_a^{-1}]_{1 \leq a \leq n} \subset \mathbb{U}$ central family Hopf torus.
- The Drinfeld double $\mathbb{U}_{\mathbb{F}} = U_{\mathbb{F}}^{\geq 0} \otimes_{\mathbb{F}} U_{\mathbb{F}}^{\leq 0}$ was considered by (Berenstein–Greenstein, 2017) and (Lacabanne, 2018); at $t_a = 1$ it gives (Drinfeld/Jimbo, 1985).
- $\mathbb{U}|_{t_a=1}$: DCP positive roots, Lusztig negative divided powers, extended Newton torus;
close to the hybrid quantum groups of (Habiro–Lê, 2016), (Situ, 2022).

Verma characterisation and Harish–Chandra theorem

- *Family Verma module* $\mathbb{M}_{\mathbb{F}}(\lambda) := \mathbb{U}_{\mathbb{F}} \otimes_{\mathbb{U}_{\mathbb{F}}^{\geq 0}[(K_a^-)^{\pm 1}]} \mathbb{Q}(q)[t_a^{\pm 1}] m_{\lambda}$,
 $e_i m_{\lambda} = 0$, $K_a^- m_{\lambda} = q^{-\lambda_a} m_{\lambda}$, $K_a^+ m_{\lambda} = t_a q^{\lambda_a} m_{\lambda}$;
 lattice $\mathbb{M}(\lambda) := \mathbb{A}_t \mathbb{U}_{\mathbb{A}}^{\leq 0} m_{\lambda}$.
- $\text{HC} : \mathbb{U}_{\mathbb{F}} \rightarrow \mathbb{U}_{\mathbb{F}}^0 = \mathbb{F}[(K_a^{\pm})^{\pm 1}]$ root-free PBW projection;
 $Y_a := q^{2(n-a)} K_a^+ (K_a^-)^{-1}$, $D := K_1^- \cdots K_n^-$; on m_{λ} :
 $\text{ev}_{\lambda}(Y_a) = t_a q^{2(\lambda_a + n - a)}$, $\text{ev}_{\lambda}(D) = q^{-\sum_a \lambda_a}$.

Theorem (Hausel–Löwit, 2026)

- 1 $\mathbb{U} = \{x \in \mathbb{U}_{\mathbb{F}} : x\mathbb{M}(\lambda) \subseteq \mathbb{M}(\lambda) \ \forall \lambda \in \mathbb{Z}^n\}$.
 - 2 $\text{HC} : Z(\mathbb{U}) \xrightarrow{\sim} \mathbb{F}[t_a^{\pm 1}, D^{\pm 1}, Y_a^{\pm 1}]^W \cap \mathbb{U} \subset \mathbb{U}_{\mathbb{F}}$.
- Proof via rational HC, Verma separation + (Drinfeld, 1990):
 $Z(\mathbb{U}_{\mathbb{F}}) \cong \mathbb{Q}(q)[t_a^{\pm 1}, D^{\pm 1}, e_1(Y), \dots, e_{n-1}(Y)]$;
 $R_{\tau} = \sum_i x_i \otimes y_i$ the canonical element of τ , $V = \Lambda_q^r \mathbb{F}^n$,
 $M_V := (1 \otimes \rho_V)(R_{\tau, 21} R_{\tau})$ commutes with $(1 \otimes \rho_V) \Delta(\mathbb{U}_{\mathbb{F}})$;
 $C_r := q^{r(n-1)} (1 \otimes \text{Tr}_V) \left[(1 \otimes \rho_V(K_{2\rho}^+)) M_V \right]$ central,
 $\text{HC}(C_r) = e_r(Y)$.

Equivariant K-theory of Gr^μ

- G complex reductive, X complex projective variety, $G \subset X$
- for $G = \mathrm{GL}_n$: $\mathrm{Gr} := \mathrm{GL}_n(\mathbb{C}((u))) / \mathrm{GL}_n(\mathbb{C}[[u]])$ *affine Grassmannian*, $\mathrm{Gr}^\mu := \mathrm{GL}_n(\mathbb{C}[[u]]) u^\mu$ *affine Schubert variety*, for μ dominant; $G^\vee = \mathrm{GL}_n(\mathbb{C})$ acts on the left, $\mathbb{C}_{\mathrm{rot}}^\times$ by $u \mapsto s^2 u$; $T^\vee \subset G^\vee$ torus, $(\mathrm{Gr}^\mu)^{T^\vee} = \mathrm{wt}(\mu) := \{\text{weights of } V^\mu\}$
- (Löwitt 2025) defines *fixed point scheme* $\mathrm{Fix}_G(X) : R \mapsto \{(g, x) \in G(R) \times X(R) \mid gx = x\}$.
develops $K_G(X; \mathbb{C}) \cong \mathbb{C}[\mathrm{Fix}_G]^G$ for a nice class of varieties
- (Löwitt 2026) proves (Bezrukavnikov 2023) conjecture:
 $K_{T^\vee \times \mathbb{C}^\times}(\mathrm{Gr}^\mu; \mathbb{C}) = \mathbb{C}[z_a^{\pm 1}, q^{\pm 1}] \langle b_{i,\ell} \rangle_{\ell > 0}^{1 \leq i \leq n}$
where $b_{i,\ell}$ are the *Bezrukavnikov classes*
- $H := T^\vee \times \mathbb{C}_{\mathrm{rot}}^\times$, $q(s) = s$; GKM description
$$K_H(\mathrm{Gr}^\mu; \mathbb{C}) \xrightarrow[\cong]{\mathrm{GKM}^\mu} \left\{ (f_\lambda)_\lambda \in \bigoplus_{\lambda \in \mathrm{wt}(\mu)} \mathbb{C}[z_a^{\pm 1}, q^{\pm 1}] : \right. \\ \left. \left(1 - \frac{z_a}{z_b} q^{(\lambda+\nu)_a - (\lambda+\nu)_b} \right) \mid (f_\lambda - f_\nu) \text{ for } \lambda - \nu \in \mathbb{Z}(\varepsilon_a - \varepsilon_b), a < b \right\}$$
- $\mathrm{GKM}^\mu(b_{1,\ell})_\lambda = \frac{\sum_a z_a^\ell - \sum_a z_a^\ell q^{2\ell\lambda_a}}{1 - q^{2\ell}} \in \mathbb{C}[z_a^{\pm 1}, q^{\pm 1}]$

Completed centre is equivariant K -theory

- $\chi_\lambda := \text{ev}_\lambda \circ \text{HC} : Z(\mathbb{U}) \rightarrow \mathbb{A}_t$, the scalar by which $Z(\mathbb{U})$ acts on $\mathbb{M}(\lambda)$. *Verma windows* $\text{wt}(\mu) \subset \mathbb{Z}^n$:

$$J_\mu := \bigcap_{\lambda \in \text{wt}(\mu)} \text{Ann}_{\mathbb{U}} \mathbb{M}(\lambda), \quad I_\mu := Z(\mathbb{U}) \cap J_\mu = \bigcap_{\lambda} \ker \chi_\lambda;$$

$$\widehat{\mathbb{U}} := \varprojlim_{\mu} \mathbb{U}/J_\mu, \quad \widehat{Z}(\mathbb{U}) := \varprojlim_{\mu} Z(\mathbb{U})/I_\mu.$$

Theorem (Hausel–Löwit, 2026)

Fixed-point restriction gives

$$\widehat{Z}(\mathbb{U}) \xrightarrow{\sim} Z(\widehat{\mathbb{U}}) \xrightarrow[\sim]{\Theta} K_H(\text{Gr}), \quad i_\lambda^* \Theta(C) = \chi_\lambda(C).$$

- $B_\ell := \Theta^{-1}(b_{1,\ell}) \in Z(\mathbb{U})$ the *Bezrukavnikov classes*; by (Löwit, 2026) $Z(\mathbb{U}_{\mathbb{F}}) = \mathbb{F}[t_a^{\pm 1}][D^{\pm 1}, B_\ell]_{\ell \in \mathbb{Z}_{>0}}$.
- Their *elementary counterparts* by Newton identities: $E_0 = 1$, $E_m = \frac{1}{m} \sum_{\ell=1}^m (-1)^{\ell-1} B_\ell E_{m-\ell}$ are actually in $Z(\mathbb{U})$.
elementary Bezrukavnikov classes

Quantum medium algebras and the affine Hecke centre

- V^μ Weyl lattice of $\mathbb{U}|_{t_a=1}$ for $\mu_1 \geq \dots \geq \mu_n \geq 0$;
 $z \in Z(\mathbb{U})$ acts on $\mathbb{M}(0) \otimes_{\mathbb{A}} V^\mu$ by $\chi_0(z)$ and by $\Delta_\mu(z)$.

$$\mathcal{Z}_q^\mu := \left(\mathbb{A}_t \left[\Delta_\mu(z) : z \in Z(\mathbb{U}) \right] \right)^W \subseteq \text{End}_{\mathbb{U}} \left(\mathbb{M}(0) \otimes_{\mathbb{A}} V^\mu \right),$$
Quantum medium algebra.
- (Orellana–Ram, 2007) On $\mathbb{M}(0) \otimes V_{\text{st}}^{\otimes d}$: $T_i := P_{i,i+1} R_{i,i+1}^{V_{\text{st}}, V_{\text{st}}}$,
 $X_1 := q^{2(n-1)} R_{21} R_{12}$, $X_{i+1} := T_i X_i T_i$; braid relations,
 $(T_i - q)(T_i + q^{-1}) = 0$, $[X_i, X_j] = 0$; they commute with $\mathbb{U}$
 $\leadsto \rho_d : \mathcal{H}_d^{\text{aff}} \rightarrow \text{End}_{\mathbb{U}} \left(\mathbb{M}(0) \otimes V_{\text{st}}^{\otimes d} \right)$ an action of the *affine Hecke algebra* $\mathcal{H}_d^{\text{aff}}$
- Bernstein centre:*

$$Z(\mathcal{H}_d^{\text{aff}}) = \mathbb{A}[X_1^{\pm 1}, \dots, X_d^{\pm 1}]^{S_d} = \mathbb{A}[e_1(X), \dots, e_{d-1}(X), e_d(X)^{\pm 1}].$$
- $$Z(\mathcal{H}_{|\mu|}^{\text{aff}}) \xrightarrow{\rho_{|\mu|}} \text{End}_{\mathbb{U}, \mathcal{H}_{|\mu|}} \left(\mathbb{M}(0) \otimes V_{\text{st}}^{\otimes |\mu|} \right) \xrightarrow{\mu\text{-cell}} \text{End}_{\mathbb{U}} \left(\mathbb{M}(0) \otimes V^\mu \right)$$

Theorem (Hausel–Löwit–Ngo, 2026)

- ① $z \mapsto (\chi_\lambda(z))_{\lambda \in \text{wt}(\mu)}$ gives

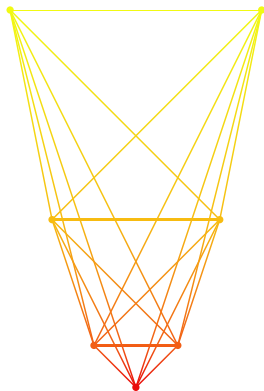
$$\mathcal{Z}_q^\mu \xrightarrow{\sim} K_{G^\vee \times \mathbb{C}_{\text{rot}}^\times}(\text{Gr}^\mu).$$

- ② $\text{Gr}^\mu \subseteq \{L \subseteq \mathbb{C}[[u]]^n : \dim \mathbb{C}[[u]]^n / L = |\mu|\}$, Q the tautological rank- $|\mu|$ quotient:

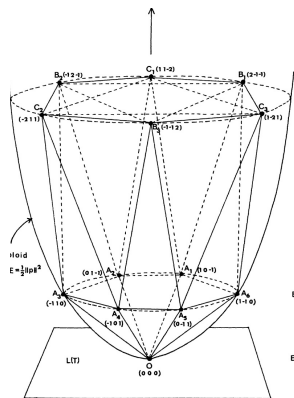
$$\begin{array}{ccccc} \mathbb{A}_t^W \otimes_{\mathbb{A}} Z(\mathcal{H}_{|\mu|}^{\text{aff}}) & \twoheadrightarrow & \mathcal{Z}_q^\mu & \xrightarrow{\sim} & K_{G^\vee \times \mathbb{C}_{\text{rot}}^\times}(\text{Gr}^\mu) \\ e_m(X) & \mapsto & \Delta_\mu(E_m) & \leftrightarrow & [\Lambda^m Q] \end{array}$$

- ③ $\mathcal{Z}_q^\mu = \mathbb{A}_t^W[E_1, \dots, E_{|\mu|}]$.

Moment polytope and GKM graph



Gr_{SL_2} (Harada–Henriques–Holm 2006)



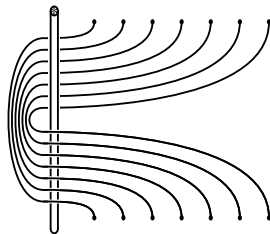
$\Omega\text{SU}(3)$ (Atiyah–Pressley 1983)

Affine braid pictures

$$T_i = \begin{array}{c} \text{Diagram with 8 vertical strands. The 4th strand from the left crosses over the 5th strand. The top of the 4th strand is labeled } ii+1. \end{array}$$

$$X_1 = \begin{array}{c} \text{Diagram with 8 vertical strands. The 1st strand from the left has a loop that crosses over itself. The top of the 1st strand is labeled } ii+1. \end{array}$$

$$X_i = \begin{array}{c} \text{Diagram with 8 vertical strands. The 1st strand from the left has a loop that crosses over itself. The top of the 1st strand is labeled } ii+1. \end{array} X_1 \cdots X_7 =$$



Connections to the Hitchin system

- *Dol–NAH*: classical medium algebra $\mathcal{M}^\mu \leftrightarrow$ upward flows (Hausel–Hitchin, 2022; Hausel, 2023–24; González, 2026):

$$H_{G^\vee}^{2*}(\mathrm{Gr}^\mu; \mathbb{C}) \cong \mathcal{M}^\mu \rightsquigarrow \text{Hecke transformations for } (W_\mu^+ \xrightarrow{h} \mathcal{A}).$$

- *Hodge–NAH*: Rees-filtered medium algebras (Hausel, 2025):

$$H_{G^\vee \times \mathbb{C}_{\mathrm{rot}}^\times}^{2*}(\mathrm{Gr}^\mu; \mathbb{C}) \cong \mathcal{Z}_\hbar^\mu \overset{?}{\rightsquigarrow} \text{Hecke transformations.}$$

- *Quantum Betti–NAH?*: q -difference NAH (Mochizuki, 2019):

$$K_{G^\vee \times \mathbb{C}_{\mathrm{rot}}^\times}(\mathrm{Gr}^\mu) \cong \mathcal{Z}_q^\mu \overset{?}{\rightsquigarrow} \text{Hecke transformations.}$$

At $q=1$: Hecke transformations for multiplicative Hitchin systems?

- *Cohomological limit*: $\mathfrak{I} := (q-1, t_a-1, e_\beta : \beta > 0) \subset \mathbb{U}$ two-sided, $\deg \mathfrak{I}/\mathfrak{I}^2 = 2$:

$$\mathbb{U}_\hbar := \mathrm{gr}_{\mathfrak{I}} \mathbb{U}, \quad Z(\mathbb{U}_{\hbar,c}) \overset{?}{\cong} H_H^{2*}(\mathrm{Gr}_c; \mathbb{Z}).$$

Tip of the iceberg of the Hitchin system



Concept: Nigel Hitchin
Design: Miguel González
Producer: Tamás Hausel

Boldog születésnapot Nigel!

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