Supplementary paper for *An Extended Cut-cell Method for Sub-Grid Liquids Tracking with Surface Tension*

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1 ALGORITHM FOR FINDING CUT CELL ISOVALUES AND ISO-SURFACES

Here, we propose a method to determine an isovalue ϕ_i for each cell i as well as the corresponding pressure samples. While the algorithm described is also an approximation, it is accurate enough to yield second-order accuracy for pressure solving, as will be numerically shown in Section 2. Following the conventions of the main paper, the algorithm is described in 2D, though it similarly generalized trivially to 3D.

Let $\Phi(\mathbf{x})$ be the signed distance function to the liquid–air interface. For each edge E_j, its centroid $\mathbf{c_j}$, and its outwards facing normal $\mathbf{n}_{E_j}$, let d_j be the distance from $\mathbf{c_j}$ to the liquid surface in the direction $-\mathbf{n}_{E_j}$. Let $m_j = \min_{w\in[0,d_j]} \Phi(\mathbf{c_j} - w\mathbf{n}_{E_j})$ and $M_j = \max_{w\in[0,d_j]} \phi(\mathbf{c_j} - w\mathbf{n}_{E_j})$. Clearly, we want the desired isovalue $\phi_i \in [m_j, M_j]$ for all edges E_j. Such a value is not guaranteed to exist, as shown in Figure 1. However, since a minute fraction of the free surface converges to plane (or line in 2D) segment, given a small enough grid spacing, such an isosurface will always exist. Furthermore, the signed distance function of a plane is linear, thus we can also approximate the signed distance function for all points on a ray by sampling several points along it and linearly interpolate the values in between. We can thus outline our method for finding the pressure samples: for all grid edges E_j, evenly sample $s + 1$ points from $\mathbf{c_j}$ to $\mathbf{c_j} - d_j\mathbf{n}_{E_j}$, including both end points. Find the maximum and minimum values of the signed distance function for all sampled points to form an interval. Find the intersection for all the intervals associated with each grid edge. We define ϕ_i to be the mean of the intersection. To find the pressure samples themselves, we use the previous samples for each grid edge and find the pressure samples using linear interpolation.

Algorithm 1 specifies the exact steps for the algorithm. Note that, since the intersection of isovalue intervals may not exist, we derive extra steps to ensure that the algorithm always outputs a meaningful value.

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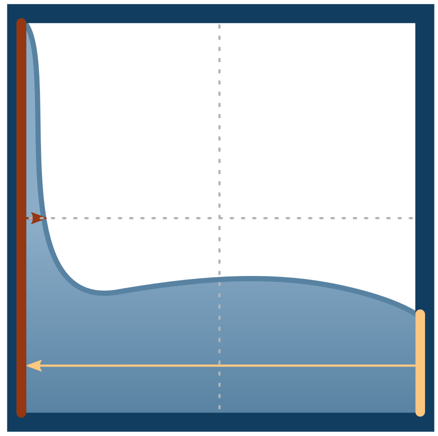


Fig. 1. Here, all isovalues along the red arrow are smaller than the isovalues along the yellow arrow. As such, finding an isoline in this cell is impossible. Note however if the cell is subdivided into four smaller cut cells along the grey dashed lines, the problem is remedied.

1.1 Considerations for 3D simulation

Our algorithm starts from the centroids of each grid edge. Since the centroid of a line segment is always its midpoint, the resulting pressure samples are always inside the liquid surface. For 3D simulations however, non-convex grid faces may result in centroids outside of the face, hence breaking the algorithm. We solve this problem by starting the algorithm from a point inside the face when a centroid outside the face is detected by exhaustively searching inside the axis-aligned bounding box of the face. In our experience, such scenarios are rare, hence having negligable impact on the overall computing time.

2 CONVERGENCE TESTS

We test the accuracy of liquid cut cells by using it as a Poisson solver to solve equations of the form

$$\nabla^2 f(x) = g(x) \quad \forall x \in D \tag{1}$$

$$f(x) = f^*(x) \quad \forall x \in \partial D \tag{2}$$

where both $g(x)$ and $f^*(x)$ are known. Here, we split the result into two cases: one where $f^*(x) = 0$, which corresponds to the case with no surface tension, and one where $f^*(x)$ can be some arbitrary value, corresponding to the case with surface tension. Furthermore, we plot the errors as a function of grid resolution to investigate the convergence of the methods.

Our treatment of the solid boundary is based on Ng et al. [1], who provided various numerical experiments to prove the second-order accuracy of the pressure values for their Neumann boundary conditions. As such, we will only test convergence of the Dirchlet boundary conditions proposed in this thesis.

The convergence test is done in a 2D domain and implemented in MATLAB. We experimented with four functions, two of which have zero at the boundary, and two of which have arbitrary values.

Algorithm 1: Pressure sample generation

Data: The centroids of each grid edge $\mathbf{c_1}, \mathbf{c_2}, \ldots, \mathbf{c_e}$, the outwards-pointing normals of each the grid edge $\mathbf{n_1}, \mathbf{n_2}, \ldots, \mathbf{n_e}$, a positive integer s, and the signed distance function of the liquid surface Φ

Result: ϕ_i, the isovalue associated with this cut cell, and scalars $d_1, d_2, \ldots, d_e$ such that the isovalue of $\mathbf{c_k} - d_k \mathbf{n_k}$ is approximately ϕ_i for each $k = 1, 2, \ldots, e$

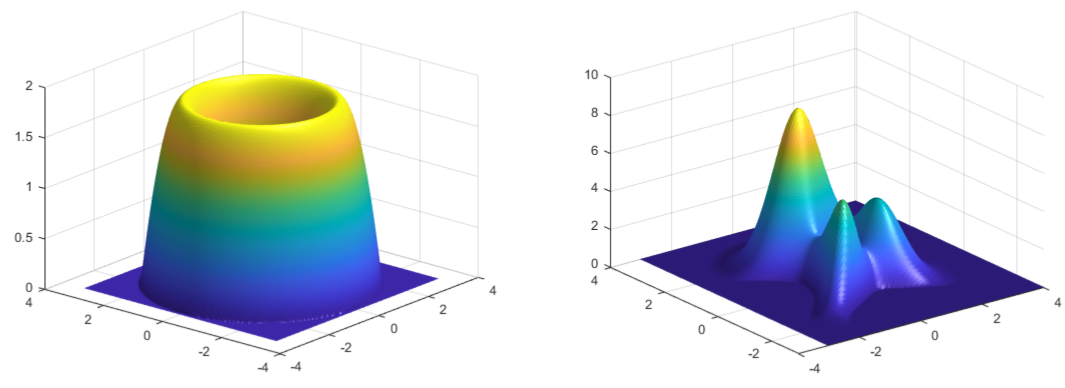
$m \leftarrow 0, M \leftarrow \infty$;
for $i \leftarrow 1$ **to** e **do**
    $d \leftarrow \min(k$ such that $c_i - k\mathbf{n_i}$ is on the liquid surface, $\Delta x)$;
    $m_i \leftarrow \infty, M_i \leftarrow 0$;
    for $j = 0$ **to** s **do**
        $r_{i,j} \leftarrow \Phi(\mathbf{c_i} - \frac{j}{s} d\mathbf{n_i})$;
        $m_i \leftarrow \min(m_i, r_{i,j})$;
        $M_i \leftarrow \max(m_i, r_{i,j})$;
    end
    $m \leftarrow \max(m, m_i)$;
    $M \leftarrow \min(M, M_i)$;
end
$\phi_i \leftarrow \frac{m+M}{2}$;
for $i \leftarrow 1$ **to** e **do**
    $d \leftarrow \min(k$ such that $c_i - k\mathbf{n_i}$ is on the liquid surface, $\Delta x)$;
    diff $\leftarrow \infty$;
    for $j = 0$ **to** $s - 1$ **do**
        if ϕ_i is between $r_{i,j}$ and $r_{i,j+1}$ **then**
            $t \leftarrow \frac{\phi_i - r_{i,j}}{r_{i,j+1} - r_{i,j}}$;
            $d_i \leftarrow \frac{j+t}{s} d$;
            break;
        else
            if $\left|\phi_i - r_{i,j}\right| < \mathit{diff}$ **then**
                $d_i \leftarrow \frac{j}{s} d$;
                diff $\leftarrow \left|\phi_i - r_{i,j}\right|$;
            end
            if $\left|\phi_i - r_{i,j+1}\right| < \mathit{diff}$ **then**
                $d_i \leftarrow \frac{j+1}{s} d$;
                diff $\leftarrow \left|\phi_i - r_{i,j+1}\right|$;
            end
        end
    end
end

2.1 Sine Function

The sine function (Figure 2a) has the form

$$f(x, y) = \sqrt{x^2 + y^2}\sin(\sqrt{x^2 + y^2})$$

with the boundary $x^2 + y^2 = \pi$.



(a) Sine function (b) Peaks function

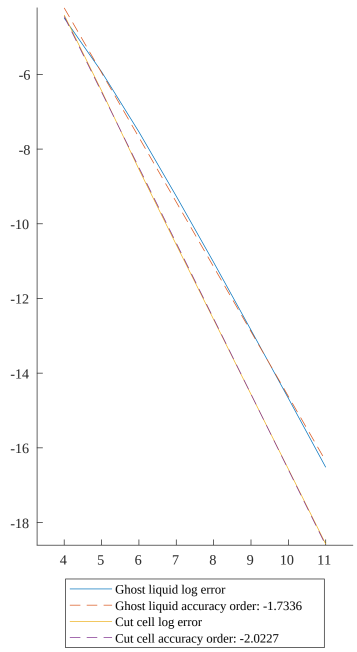
Fig. 2. Functions with zero boundaries

Table 1. Sine Function, Ghost Liquid

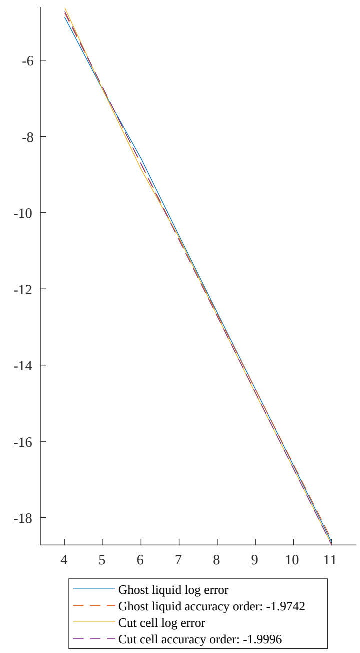
	16×16	32×32	64×64	128×128	256×256	512×512	1024×1024	2048×2048
Interior Values, L_∞	0.04448349	0.016416235	0.005377668	0.001629738	0.000477745	0.00013582	3.83E-05	1.06E-05
Boundary values, L_∞	0.034110824	0.0091037	0.002622942	0.000641416	0.000158681	3.94E-05	9.83E-06	2.45E-06
Interior gradients, L_∞	0.101348244	0.051522125	0.025636128	0.01275109	0.006354695	0.003171643	0.001584335	0.000791788
Boundary gradients, L_∞	0.07245751	0.039204127	0.01796989	0.019712104	0.009086708	0.004652856	0.002392643	0.004347745
Interior Values, RMSE	0.026423193	0.006852558	0.001839952	0.000466222	0.000117002	2.82E-05	7.02E-06	1.72E-06
Boundary values, RMSE	0.027024588	0.006265309	0.001712701	0.000428066	0.000102154	2.49E-05	5.99E-06	1.50E-06
Interior gradients, RMSE	0.017136615	0.004188861	0.001154237	0.000311	8.18E-05	2.26E-05	6.29E-06	1.81E-06
Boundary gradients, RMSE	0.042139186	0.013465727	0.007621484	0.004115123	0.001956629	0.001026071	0.000504796	0.000296977
Order (Interior Values, L_∞)		1.438146681	1.61007079	1.722340287	1.770326483	1.814550798	1.826308458	1.848300036
Order (Boundary values, L_∞)		1.905704602	1.795267098	2.031853027	2.015128499	2.009051362	2.003321784	2.007229215
Order (Interior gradients, L_∞)		0.976057102	1.007013719	1.007557747	1.004725939	1.002592575	1.001352853	1.000690467
Order (Boundary gradients, L_∞)		0.886129688	1.125423946	-0.133500208	1.117252179	0.965641115	0.959511742	-0.861662566
Order (Interior Values, RMSE)		1.947090284	1.896974129	1.980578996	1.994489557	2.050310498	2.008429472	2.030623001
Order (Boundary values, RMSE)		2.108815086	1.871112397	2.000366983	2.067095291	2.039357674	2.051666501	1.996427709
Order (Interior gradients, RMSE)		2.032452317	1.859617999	1.891952409	1.927076696	1.857423004	1.843234335	1.799590125
Order (Boundary gradients, RMSE)		1.645870333	0.821148336	0.889136305	1.072565591	0.931238712	1.023357989	0.765350291

Table 2. Sine Function, Cut Cells

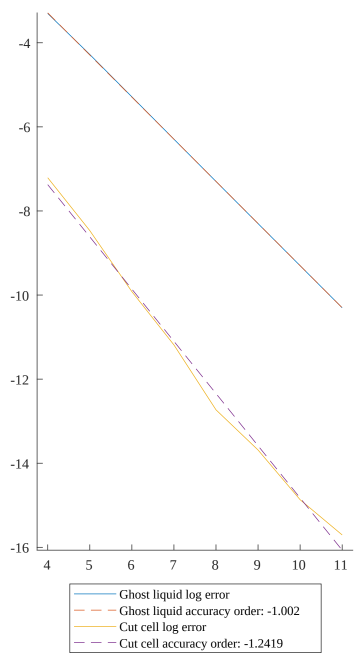
	16×16	32×32	64×64	128×128	256×256	512×512	1024×1024	2048×2048
Interior Values, L_∞	0.047093424	0.01154217	0.00268607	0.000661703	0.000165165	4.12E-05	1.03E-05	2.56E-06
Boundary values, L_∞	0.040609905	0.009133814	0.002160903	0.000620831	0.00015002	3.66E-05	9.32E-06	2.34E-06
Interior gradients, L_∞	0.006756403	0.002824745	0.001033865	0.000429154	0.000147236	7.59E-05	3.34E-05	1.87E-05
Boundary gradients, L_∞	0.064837063	0.035474949	0.015476325	0.008025816	0.00390648	0.001975945	0.001155679	0.000678465
Interior Values, RMSE	0.034883497	0.008741432	0.002132558	0.000528105	0.000132918	3.27E-05	8.21E-06	2.04E-06
Boundary values, RMSE	0.022118888	0.00452675	0.00136157	0.000322288	8.21E-05	1.86E-05	4.88E-06	1.19E-06
Interior gradients, RMSE	0.002660204	0.001068147	0.000273197	7.29E-05	1.80E-05	5.12E-06	1.35E-06	3.68E-07
Boundary gradients, RMSE	0.030903558	0.013058125	0.005970136	0.002752269	0.001300609	0.000657581	0.000318839	0.000166224
Order (Interior Values, L_∞)		2.028611141	2.103345717	2.021240382	2.002279465	2.003294773	2.001242872	2.006650209
Order (Boundary values, L_∞)		2.1525424	2.079582756	1.799362783	2.049043118	2.03620302	1.972228772	1.993536601
Order (Interior gradients, L_∞)		1.258134621	1.450072872	1.268479427	1.543369457	0.95598802	1.182149345	0.838070495
Order (Boundary gradients, L_∞)		0.87001814	1.196737709	0.947342971	1.038778887	0.983326337	0.77380238	0.768395035
Order (Interior Values, RMSE)		1.996603147	2.035284411	2.013689183	1.990284519	2.021150282	1.995829114	2.007166194
Order (Boundary values, RMSE)		2.288731334	1.733204576	2.078850539	1.972032014	2.14442602	1.928355984	2.034320713
Order (Interior gradients, RMSE)		1.316426135	1.96709491	1.906007954	2.017637604	1.814192532	1.923721483	1.875692245
Order (Boundary gradients, RMSE)		1.242825157	1.129112026	1.117142363	1.08143407	0.983946302	1.044341875	0.939694973



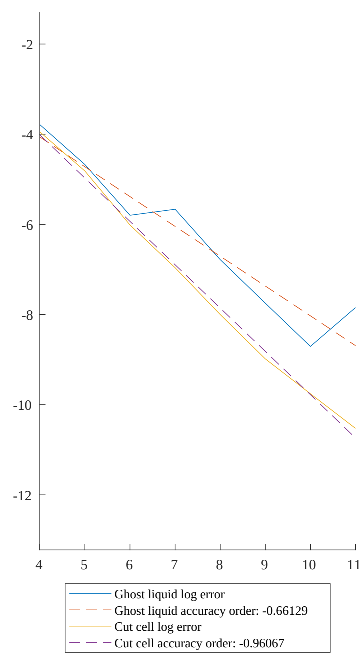
(a) Accuracy of pressure values in non-cut cells



(b) Accuracy of pressure values in cut cells

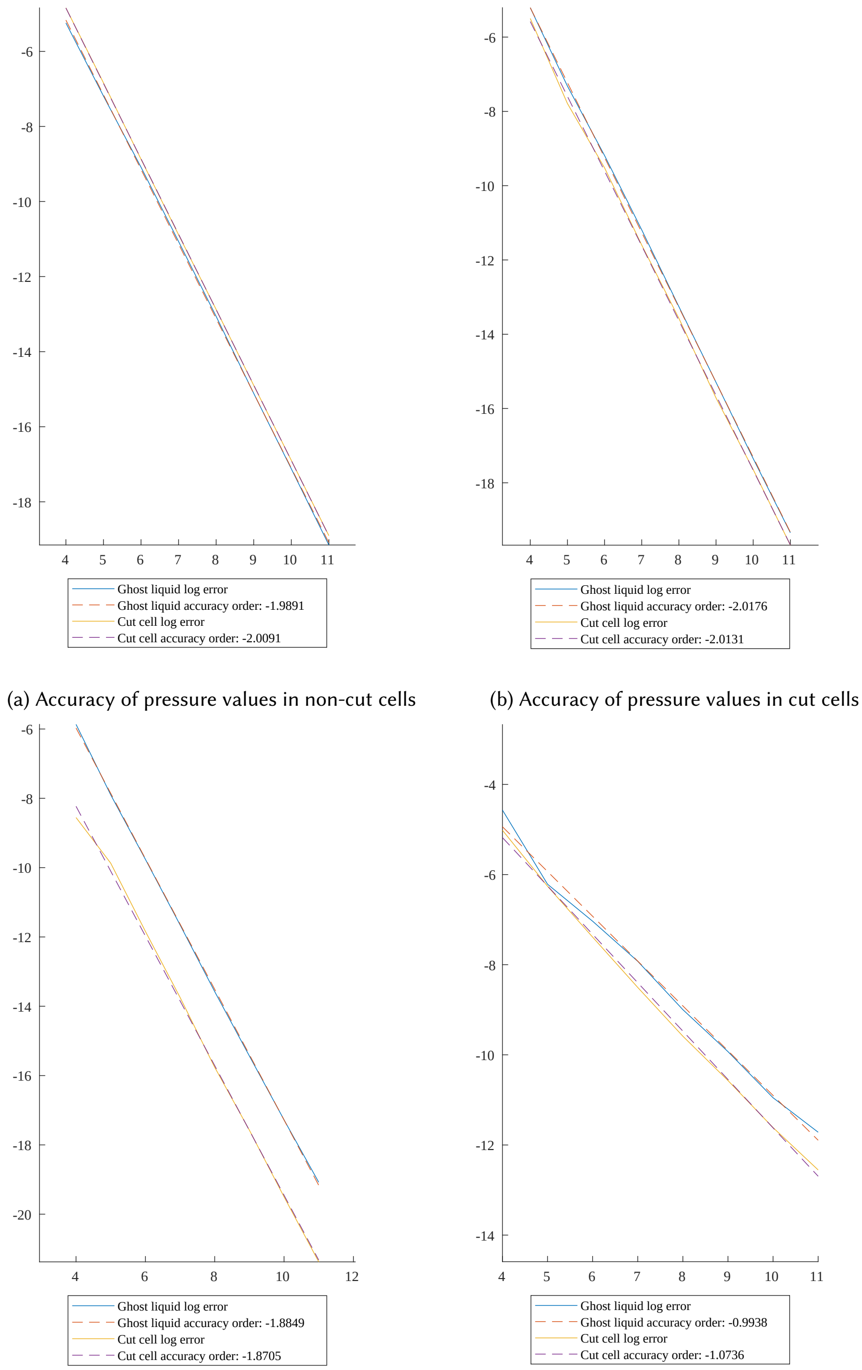


(c) Accuracy of gradient values in non-cut cells



(d) Accuracy of gradient values in cut cells

Fig. 3. Convergence of error in the L_∞, sine function



(a) Accuracy of pressure values in non-cut cells

(b) Accuracy of pressure values in cut cells

(c) Accuracy of gradient values in non-cut cells

(d) Accuracy of gradient values in cut cells

Fig. 4. Convergence of error (root-mean-square error), sine function

Table 3. Peaks Function, Ghost Fluid

	16×16	32×32	64×64	128×128	256×256	512×512	1024×1024	2048×2048
Interior Values, L_∞	0.344537811	0.084370515	0.021193987	0.005310682	0.001314265	0.000338491	8.76E-05	2.24E-05
Boundary values, L_∞	0.285079398	0.065049595	0.016192204	0.004431129	0.001284512	0.000338491	8.76E-05	2.24E-05
Interior gradients, L_∞	0.122718017	0.046323431	0.019895127	0.008725887	0.004574047	0.002205107	0.001136743	0.000573704
Boundary gradients, L_∞	1.305084786	1.212390808	0.390395427	0.266838288	0.159824718	0.07980462	0.041436632	0.024448837
Interior Values, RMSE	0.09367779	0.022374803	0.00550106	0.001353577	0.000336142	8.49E-05	2.12E-05	5.32E-06
Boundary values, RMSE	0.071411675	0.011190401	0.003210646	0.000839588	0.000207383	5.12E-05	1.32E-05	3.31E-06
Interior gradients, RMSE	0.052531945	0.013215217	0.003727676	0.000996931	0.000274735	7.38E-05	1.96E-05	5.47E-06
Boundary gradients, RMSE	0.367505331	0.18322215	0.073058556	0.044726008	0.020186876	0.011226202	0.005376592	0.002611296
Order (Interior Values, L_∞)		2.02985151	1.993083894	1.996686057	2.014641274	1.95706776	1.950286569	1.967735906
Order (Boundary values, L_∞)		2.131751811	2.006240671	1.869553166	1.786453431	1.924032881	1.950286569	1.967735906
Order (Interior gradients, L_∞)		1.405533055	1.21932697	1.189041373	0.931830802	1.052622445	0.955942738	0.986528395
Order (Boundary gradients, L_∞)		0.106288716	1.634846762	0.548970463	0.739475137	1.001946381	0.945565535	0.761140884
Order (Interior Values, RMSE)		2.065832061	2.02409332	2.022932458	2.009634159	1.985802649	2.002049049	1.994684666
Order (Boundary values, RMSE)		2.67389823	1.801326061	1.935110245	2.017380527	2.017181429	1.957692739	1.996344319
Order (Interior gradients, RMSE)		1.990994907	1.825851719	1.902711053	1.859452537	1.897290061	1.912565732	1.840972191
Order (Boundary gradients, RMSE)		1.004171256	1.326468773	0.707939249	1.147696338	0.846547731	1.06210599	1.04192578

Table 4. Peaks Function, Cut Cells

	16×16	32×32	64×64	128×128	256×256	512×512	1024×1024	2048×2048
Interior Values, L_∞	0.64007989	0.088248709	0.022030747	0.005559601	0.001483982	0.000385236	9.81E-05	2.48E-05
Boundary values, L_∞	1.869472756	0.121857962	0.03205369	0.008515087	0.001925473	0.00053515	0.000129835	3.38E-05
Interior gradients, L_∞	0.612498347	0.092454172	0.039699055	0.015656947	0.005569445	0.001824432	0.00097955	0.000530617
Boundary gradients, L_∞	4.280202444	1.134027647	0.69486986	0.375488745	0.200204876	0.109844811	0.052072063	0.026848817
Interior Values, RMSE	0.306822513	0.02591487	0.006200104	0.001496803	0.000368419	9.14E-05	2.27E-05	5.68E-06
Boundary values, RMSE	0.478346216	0.022230739	0.004607534	0.001134016	0.00026878	6.75E-05	1.67E-05	4.21E-06
Interior gradients, RMSE	0.206672922	0.020333766	0.005316656	0.001435772	0.000370915	9.42E-05	2.41E-05	6.29E-06
Boundary gradients, RMSE	0.905156835	0.255868838	0.136386102	0.066828435	0.032608742	0.016547341	0.00821027	0.004113291
Order (Interior Values, L_∞)		2.858604896	2.002056751	1.986465138	1.905508034	1.945657986	1.97398665	1.983642849
Order (Boundary values, L_∞)		3.939359026	1.926638168	1.912397214	2.144808539	1.847198834	2.043266551	1.939855791
Order (Interior gradients, L_∞)		2.72789562	1.219633774	1.342301723	1.491197478	1.610086031	0.89725625	0.884448252
Order (Boundary gradients, L_∞)		1.916223222	0.706641102	0.887973142	0.907292553	0.866010393	1.076885255	0.955651048
Order (Interior Values, RMSE)		3.565552286	2.063415855	2.050407752	2.02246626	2.011357844	2.006569031	2.00160543
Order (Boundary values, RMSE)		4.42742736	2.270489197	2.022553474	2.076944438	1.993957872	2.013460907	1.988938081
Order (Interior gradients, RMSE)		3.345400041	1.935286351	1.888692843	1.952665922	1.976677607	1.96462868	1.941599693
Order (Boundary gradients, RMSE)		1.822763334	0.907707815	1.029162644	1.035203311	0.978659363	1.011097864	0.99713667

2.2 Peaks

The peaks function (Figure 2b) is a combination of three Gaussians. Precisely, it has the form:

$$f(x,y) = 3e^{-(y+1)^2-x^2}(x-1)^2 - \frac{e^{-(x+1)^2-y^2}}{3} + e^{-x^2-y^2}(10x^3 - 2x + 10y^5 + 2) - 0.03 \tag{3}$$

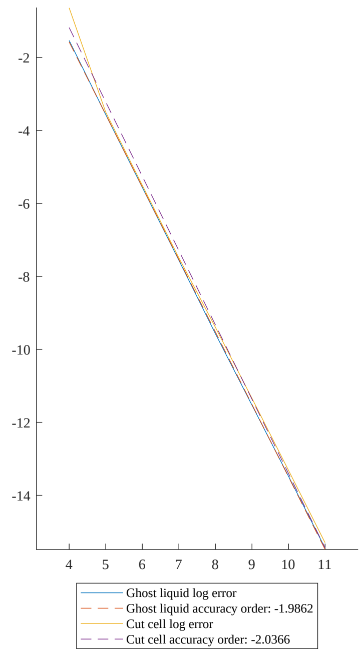
The interior is then define by $f(x,y) > 0$

2.3 Parabola

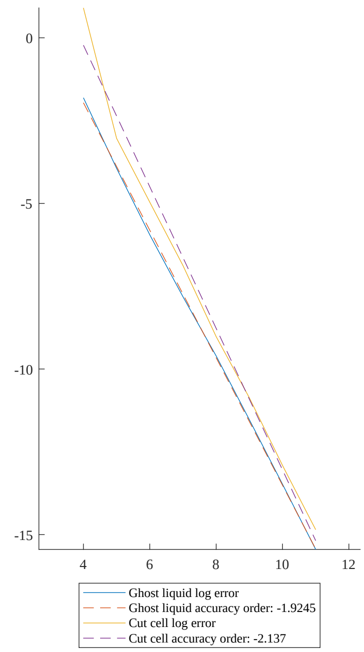
The parabola function (Figure 7a) has the form

$$f(x,y) = (x-1)^2 + (y-1)^2$$

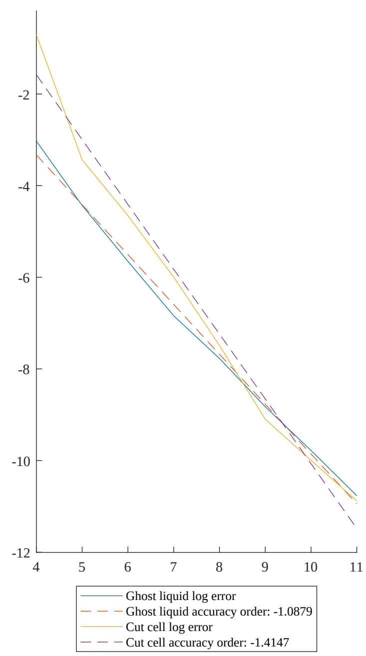
We define non-zero boundaries for this function, which gives more flexibility with the boundary choice. Here we use a circular boundary $\sqrt{x^2+y^2} = \pi$



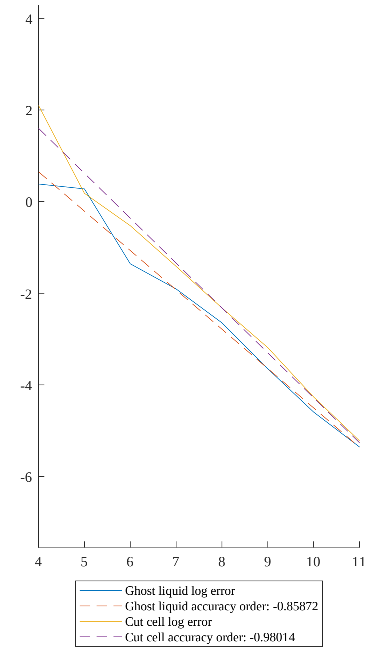
(a) Accuracy of pressure values in non-cut cells



(b) Accuracy of pressure values in cut cells

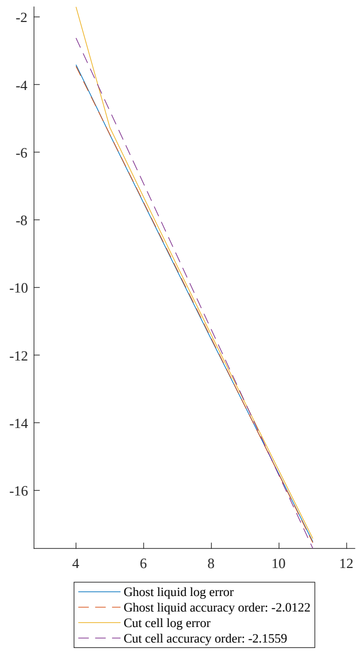


(c) Accuracy of gradient values in non-cut cells

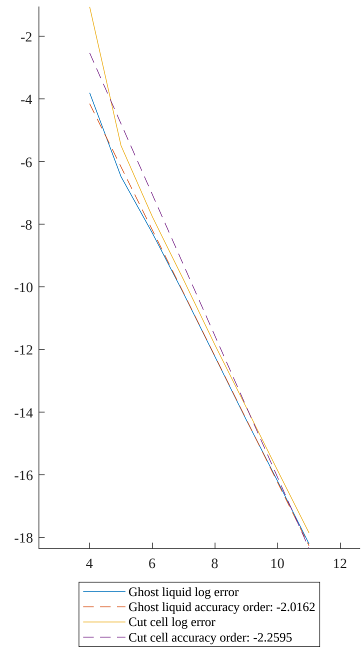


(d) Accuracy of gradient values in cut cells

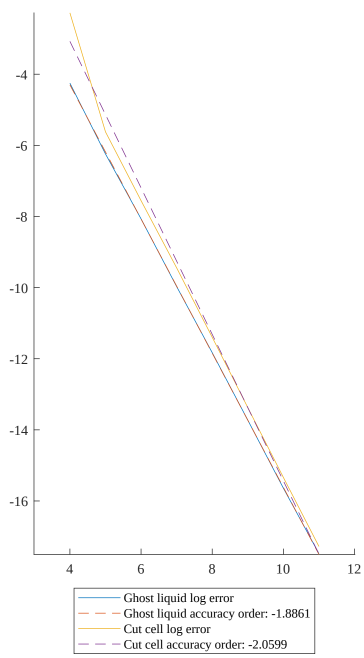
Fig. 5. Convergence of error in the L_∞, peaks function



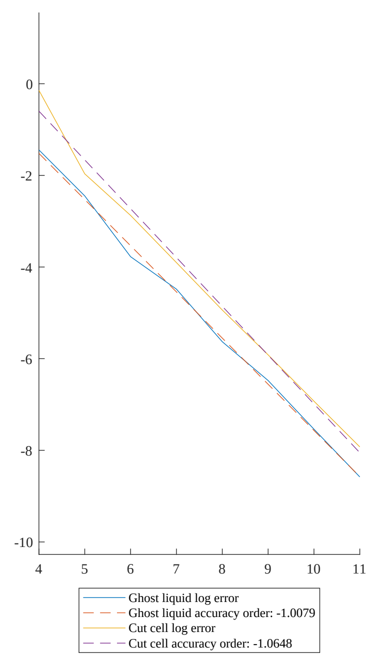
(a) Accuracy of pressure values in non-cut cells



(b) Accuracy of pressure values in cut cells

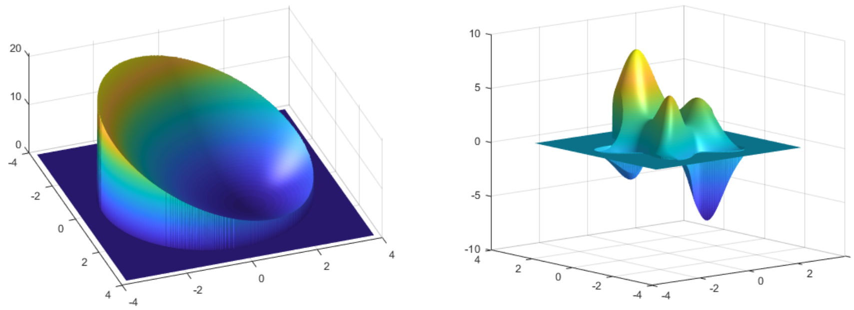


(c) Accuracy of gradient values in non-cut cells



(d) Accuracy of gradient values in cut cells

Fig. 6. Convergence of error (root-mean-square error), peaks function



(a) Parabola function

(b) Peaks2 function

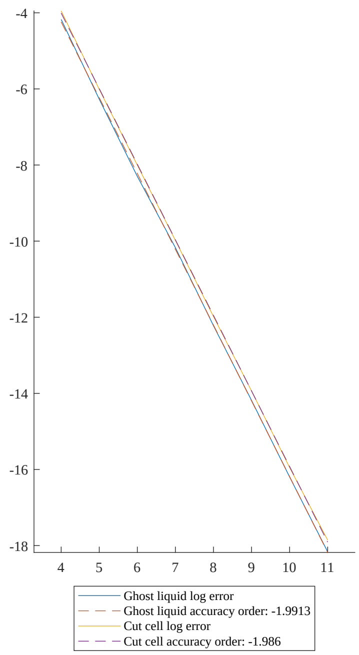
Fig. 7. Functions with non-zero boundaries

Table 5. Parabola Function, Ghost Liquid

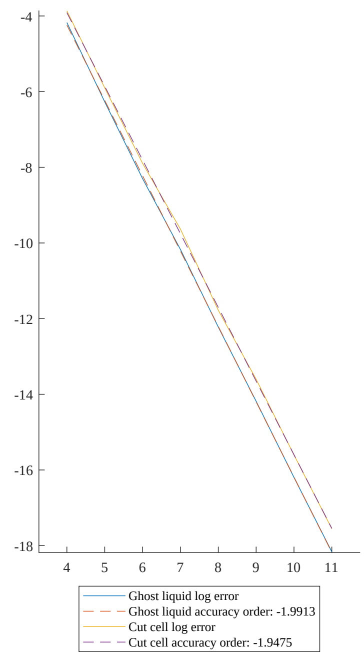
	16 × 16	32 × 32	64 × 64	128 × 128	256 × 256	512 × 512	1024 × 1024	2048 × 2048
Interior Values, L_∞	0.055398823	0.012953586	0.003176	0.000866785	0.000209305	5.37E-05	1.33E-05	3.40E-06
Boundary values, L_∞	0.055398823	0.012953586	0.003176	0.000866785	0.000209305	5.37E-05	1.33E-05	3.40E-06
Interior gradients, L_∞	0.004118294	0.005143016	0.003025947	0.000996681	0.000607187	0.000313314	1.58E-04	8.90E-05
Boundary gradients, L_∞	0.507861871	0.292325386	0.17471143	0.103808396	0.049568146	0.026357512	0.014082354	0.00694547
Interior Values, RMSE	0.044161221	0.009982645	0.002238272	0.000592986	0.000146673	3.69E-05	9.18E-06	2.28E-06
Boundary values, RMSE	0.045625791	0.009765606	0.00235799	0.000613532	0.000155655	3.88E-05	9.56E-06	2.42E-06
Interior gradients, RMSE	0.001851064	0.001165104	0.000367779	9.48E-05	3.94E-05	1.22E-05	4.62E-06	1.62E-06
Boundary gradients, RMSE	0.218924427	0.133948796	0.066339073	0.036332215	0.014858917	0.006806459	0.003646239	0.001737031
Order (Interior Values, L_∞)		2.096503759	2.028068589	1.873465483	2.050068115	1.962183645	2.012235383	1.967935144
Order (Boundary values, L_∞)		2.096503759	2.028068589	1.873465483	2.050068115	1.962183645	2.012235383	1.967935144
Order (Interior gradients, L_∞)		-0.320567806	0.765227576	1.602183212	0.714991412	0.954530009	0.985055444	0.830071582
Order (Boundary gradients, L_∞)		0.796861045	0.742601127	0.751050853	1.066437945	0.911199121	0.904325614	1.019744286
Order (Interior Values, RMSE)		2.145286032	2.157036569	1.916316738	2.015389124	1.991822967	2.006838466	2.010051051
Order (Boundary values, RMSE)		2.224068077	2.050151838	1.942348356	1.978782638	2.005092347	2.020617797	1.982665075
Order (Interior gradients, RMSE)		0.667895949	1.663549331	1.956028477	1.26751026	1.689900739	1.401320126	1.515576533
Order (Boundary gradients, RMSE)		0.708751324	1.013750844	0.868609553	1.289920322	1.126352717	0.90049515	1.069785892

Table 6. Parabola Function, Cut Cells

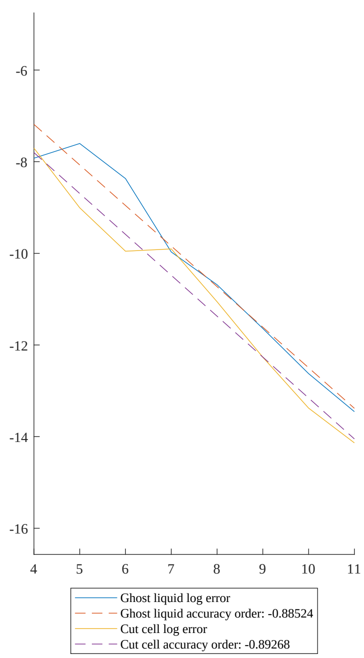
	16 × 16	32 × 32	64 × 64	128 × 128	256 × 256	512 × 512	1024 × 1024	2048 × 2048
Interior Values, L_∞	0.064711337	0.015509518	0.003898536	0.000989883	0.000247393	6.38E-05	1.58E-05	4.22E-06
Boundary values, L_∞	0.068600692	0.016740948	0.004197769	0.001258261	0.00028497	8.04E-05	2.01E-05	5.20E-06
Interior gradients, L_∞	0.004796879	0.001945629	0.0010083	0.001043902	0.000469717	0.000202819	9.39E-05	5.55E-05
Boundary gradients, L_∞	0.200837316	0.108923095	0.042418499	0.026054264	0.014367444	0.00711453	0.003645565	0.001844515
Interior Values, RMSE	0.062200196	0.014689722	0.00370071	0.00089429	0.000223497	5.63E-05	1.40E-05	3.52E-06
Boundary values, RMSE	0.030867702	0.006851116	0.001988446	0.000551262	0.000117588	3.08E-05	7.39E-06	1.86E-06
Interior gradients, RMSE	0.001329837	0.000283726	0.000122949	5.62E-05	1.65E-05	5.29E-06	1.74E-06	6.58E-07
Boundary gradients, RMSE	0.070628747	0.035076961	0.013556135	0.007635555	0.00404165	0.001828815	0.000930992	0.000465138
Order (Interior Values, L_∞)		2.060864624	1.992149482	1.97760234	2.000452116	1.955414208	2.011649928	1.907713599
Order (Boundary values, L_∞)		2.034841913	1.995686456	1.738191575	2.142549327	1.824971021	2.001165991	1.951055933
Order (Interior gradients, L_∞)		1.301859342	0.948311717	-0.050060757	1.152123003	1.211600633	1.110345945	0.758755398
Order (Boundary gradients, L_∞)		0.882717468	1.360544395	0.703174053	0.858716142	1.013962977	0.964626208	0.982900474
Order (Interior Values, RMSE)		2.082112022	1.988933102	2.048987988	2.000485851	1.989483615	2.00819279	1.989258824
Order (Boundary values, RMSE)		2.17168722	1.784697305	1.850830456	2.229001709	1.93422491	2.05761299	1.989518933
Order (Interior gradients, RMSE)		2.228680562	1.20643155	1.128241925	1.765858927	1.645497765	1.60673479	1.400387775
Order (Boundary gradients, RMSE)		1.009731738	1.371577818	0.828141091	0.917788684	1.144035026	0.974069161	1.001110311



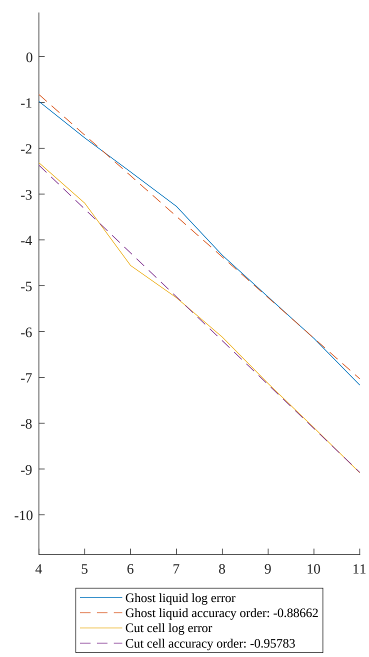
(a) Accuracy of pressure values in non-cut cells



(b) Accuracy of pressure values in cut cells

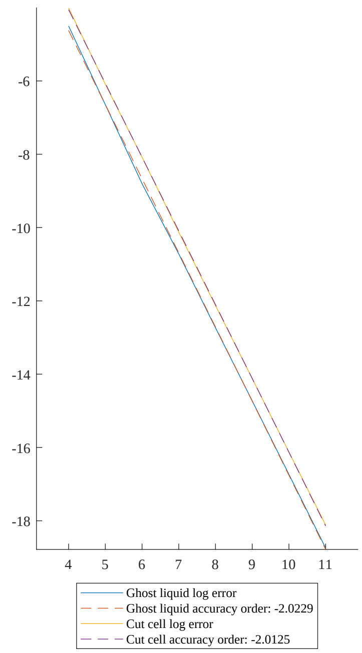


(c) Accuracy of gradient values in non-cut cells

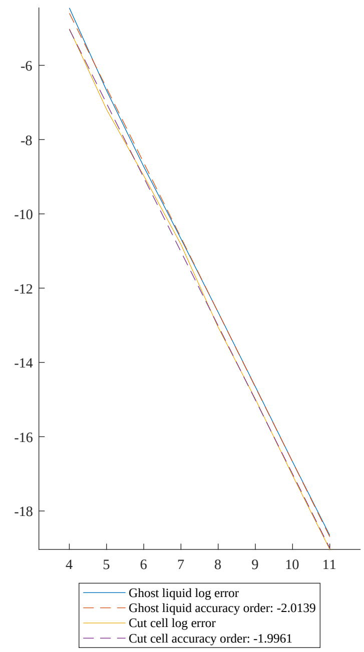


(d) Accuracy of gradient values in cut cells

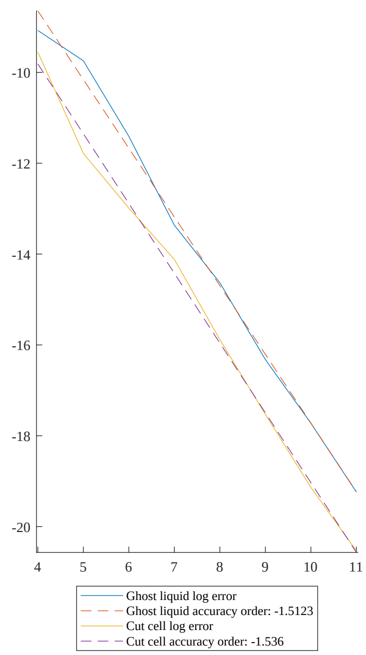
Fig. 8. Convergence of error in the L_∞, parabola function



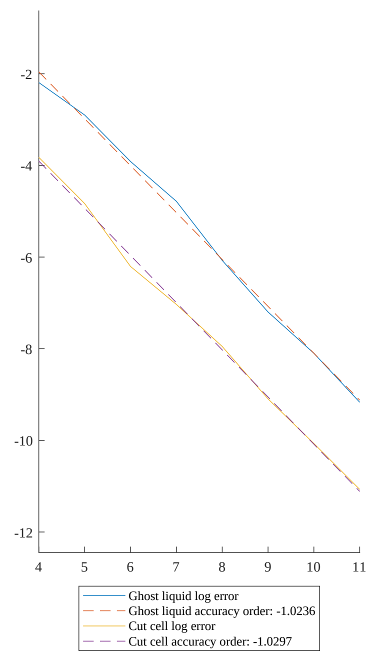
(a) Accuracy of pressure values in non-cut cells



(b) Accuracy of pressure values in cut cells



(c) Accuracy of gradient values in non-cut cells



(d) Accuracy of gradient values in cut cells

Fig. 9. Convergence of error (root-mean-square error), parabola function

Table 7. Peaks2 Function, Ghost Fluid

	16×16	32×32	64×64	128×128	256×256	512×512	1024×1024	2048×2048
16×16	32×32	64×64	128×128	256×256	512×512	1024×1024	2048×2048	
Interior Values, L_∞	0.37484148	0.073667901	0.023025503	0.006419121	0.001600754	0.000433903	0.000115786	3.02E-05
Boundary values, L_∞	0.37484148	0.071124396	0.023025503	0.006419121	0.001600754	0.000433903	0.000115786	3.02E-05
Interior gradients, L_∞	0.129150181	0.045408768	0.016330533	0.006949313	0.00392662	0.00213303	0.001461465	0.000863778
Boundary gradients, L_∞	1.492050323	1.162744882	0.734433865	0.326242335	1.368500731	0.090063339	0.391595861	0.02282262
Interior Values, RMSE	0.111021144	0.022892096	0.005442641	0.00139316	0.00034086	8.44E-05	2.15E-05	5.32E-06
Boundary values, RMSE	0.113923654	0.020435871	0.005185014	0.001517397	0.000357886	8.38E-05	2.29E-05	5.61E-06
Interior gradients, RMSE	0.06434406	0.014709906	0.003974045	0.001119632	0.000300407	8.32E-05	2.35E-05	6.78E-06
Boundary gradients, RMSE	0.520121619	0.261234644	0.116026818	0.056770285	0.063470974	0.014665656	0.010308577	0.003581023
Order (Interior Values, L_∞)		2.347172559	1.677803471	1.842784982	2.003624489	1.883307552	1.905909196	1.938875747
Order (Boundary values, L_∞)		2.397864201	1.62711183	1.842784982	2.003624489	1.883307552	1.905909196	1.938875747
Order (Interior gradients, L_∞)		1.508006883	1.475398977	1.232629731	0.823582465	0.880383311	0.545489127	0.758683071
Order (Boundary gradients, L_∞)		0.359761605	0.662830101	1.170688579	-2.068580295	3.925512435	-2.120353651	4.100829197
Order (Interior Values, RMSE)		2.277913022	2.072470752	1.96594555	2.031108993	2.013477275	1.972725637	2.014217513
Order (Boundary values, RMSE)		2.478891679	1.978684091	1.772749236	2.08402849	2.094555967	1.872349408	2.028956835
Order (Interior gradients, RMSE)		2.129018902	1.888107974	1.827584012	1.898033264	1.851667541	1.822925608	1.795084852
Order (Boundary gradients, RMSE)		0.993502772	1.170887933	1.031250415	-0.160960995	2.113655383	0.508596379	1.525401467

Table 8. Peaks2 Function, Cut Cells

	16×16	32×32	64×64	128×128	256×256	512×512	1024×1024	2048×2048
Interior Values, L_∞	0.320142826	0.141564591	0.029024769	0.008122763	0.002182695	0.000583506	0.000154253	3.96E-05
Boundary values, L_∞	0.752791451	0.254186933	0.042534908	0.01285248	0.003614913	0.000801943	0.000277335	5.51E-05
Interior gradients, L_∞	0.31965467	0.148708243	0.032829487	0.01831229	0.00559061	0.003395869	0.001560535	0.000974679
Boundary gradients, L_∞	2.983026697	1.832628898	1.372826284	0.689201541	0.435177467	0.190747673	0.115778416	0.058789311
Interior Values, RMSE	0.160085246	0.046812272	0.006756071	0.001676223	0.000424823	1.06E-04	2.65E-05	6.64E-06
Boundary values, RMSE	0.171409308	0.046741068	0.006114561	0.001630962	0.000388442	9.62E-05	2.48E-05	6.12E-06
Interior gradients, RMSE	0.13014847	0.045595007	0.007281765	0.00180272	0.000464488	0.000123414	3.27E-05	9.06E-06
Boundary gradients, RMSE	0.582905915	0.299895544	0.170438132	0.077426484	0.0388491	0.019241331	0.009470287	0.004778182
Order (Interior Values, L_∞)		1.177255223	2.286103971	1.837242147	1.895860029	1.903291629	1.919444165	1.961147339
Order (Boundary values, L_∞)		1.566360376	2.57917064	1.726600531	1.830014021	2.172388905	1.53186796	2.332010548
Order (Interior gradients, L_∞)		1.104029552	2.179420524	0.842179966	1.711734641	0.719224681	1.121740732	0.679040733
Order (Boundary gradients, L_∞)		0.702862215	0.416765594	0.994151248	0.663322069	1.189938405	0.720299132	0.977740559
Order (Interior Values, RMSE)		1.773881654	2.792630313	2.010970528	1.980279506	2.000526046	1.99982114	1.998730834
Order (Boundary values, RMSE)		1.874682855	2.934369866	1.906525369	2.069953372	2.013829641	1.95829685	2.016134922
Order (Interior gradients, RMSE)		1.513210612	2.646515801	2.014113161	1.956463279	1.912129019	1.915335554	1.851799211
Order (Boundary gradients, RMSE)		0.958802956	0.815211945	1.138349104	0.994945949	1.013672593	1.022728515	0.986946288

2.4 Peaks with non-zero boundaries

The peaks function with non-zero boundaries (Figure 7b, *Peaks2* for short) is identical to the peaks function, except the extra −0.03 is removed:

$$f(x, y) = 3e^{-(y+1)^2-x^2}(x-1)^2 - \frac{e^{-(x+1)^2-y^2}}{3} + e^{-x^2-y^2}(10x^3 - 2x + 10y^5 + 2)$$

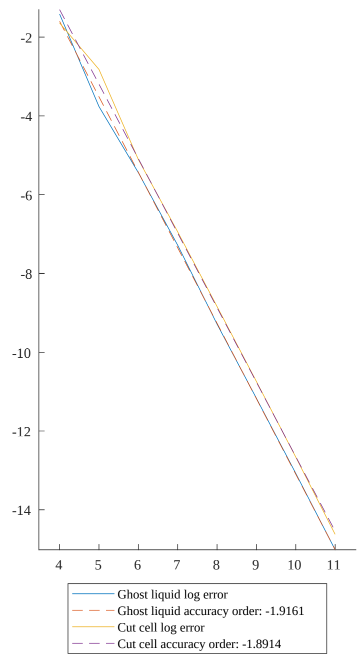
The boundary is described in polar coordinates:

$$r = 2 + \frac{1}{2}\sin(5\theta)$$

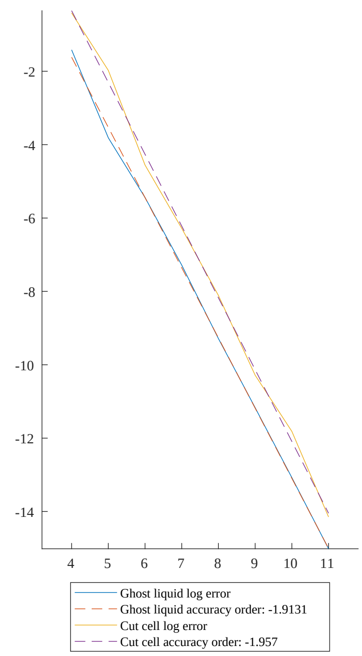
where $r = \sqrt{x^2 + y^2}$, and θ is the angle from the positive x-axis of (x, y).

2.5 Convergence test with constant iso-value

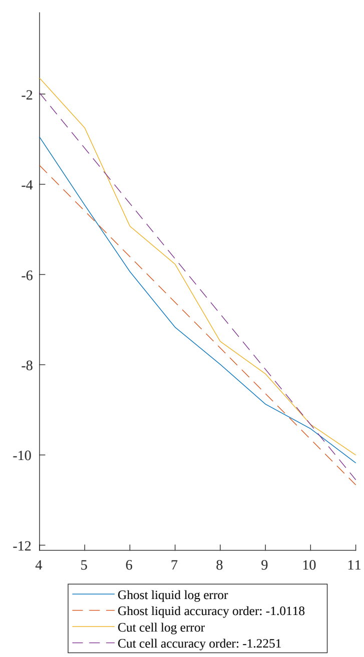
In our method, we vary ϕ_i for each cut cell. In Table 9, Figure 12, and Figure 13, we solve the parabola function with cut cells, except this time we assign the same ϕ_i for all the cells. As shown in the graphs, this drastically deteriorates the accuracy of the solution.



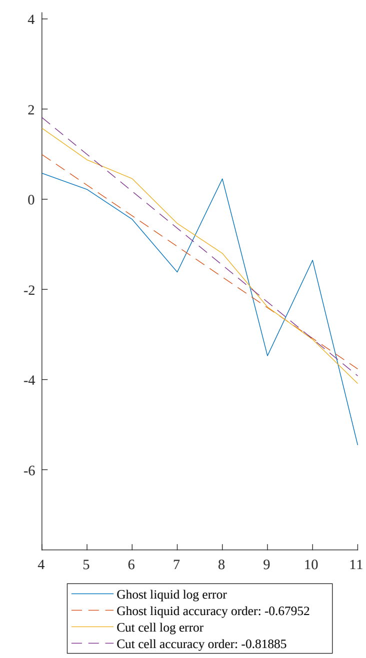
(a) Accuracy of pressure values in non-cut cells



(b) Accuracy of pressure values in cut cells

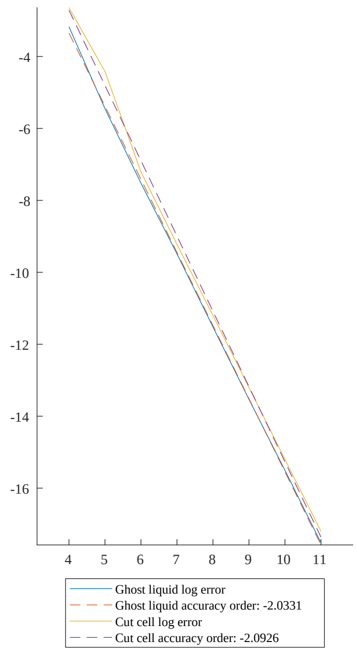


(c) Accuracy of gradient values in non-cut cells

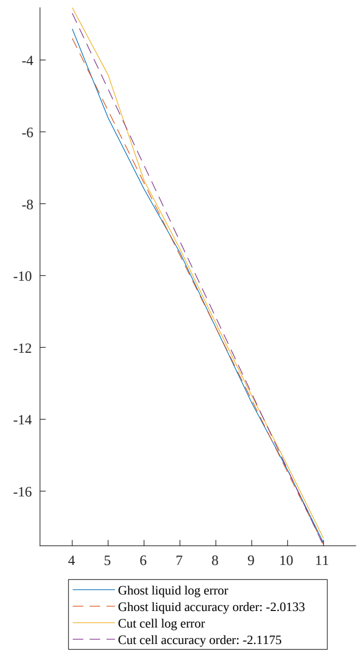


(d) Accuracy of gradient values in cut cells

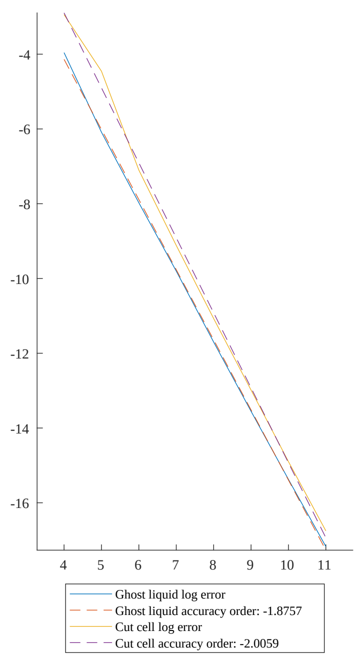
Fig. 10. Convergence of error in the L_∞, peaks2 function



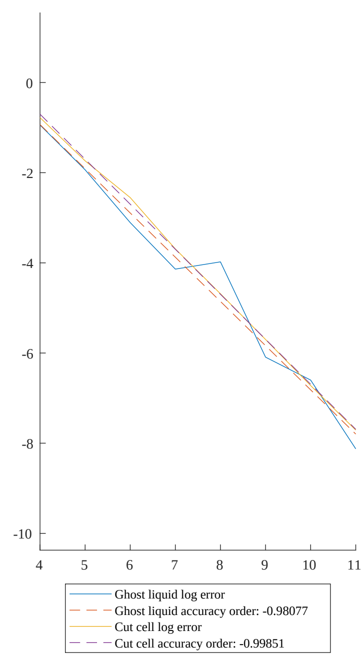
(a) Accuracy of pressure values in non-cut cells



(b) Accuracy of pressure values in cut cells



(c) Accuracy of gradient values in non-cut cells

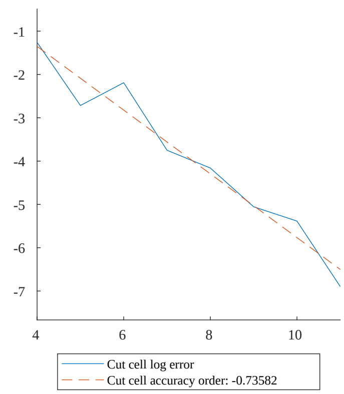


(d) Accuracy of gradient values in cut cells

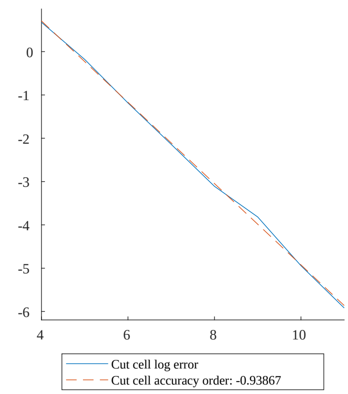
Fig. 11. Convergence of error (root-mean-square error), peaks2 function

Table 9. Parabola Function, Cut Cells with uniform isovalue

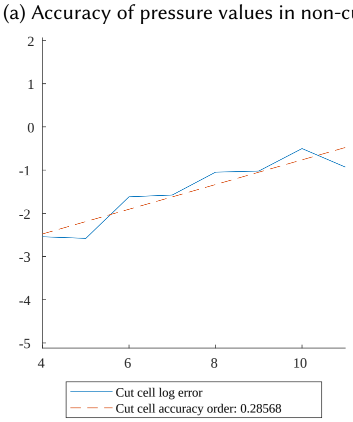
	16×16	32×32	64×64	128×128	256×256	512×512	1024×1024	2048×2048
Interior Values, L_∞	0.417826944	0.152113726	0.219081275	0.07432806	0.055935263	0.030087635	0.023928928	0.008391485
Boundary values, L_∞	1.599145533	0.882525899	0.439866798	0.226764273	0.115849892	0.070932373	0.03252445	0.01653108
Interior gradients, L_∞	0.171618846	0.167219124	0.325667725	0.33517099	0.483583604	0.492247667	0.706199115	0.524839699
Boundary gradients, L_∞	2.897038053	3.861814905	3.78203531	3.543851989	3.554993039	3.797362658	3.795003654	3.645818404
Interior Values, RMSE	0.24276541	0.09375683	0.067209041	0.026627535	0.013008612	0.007344908	0.00397918	0.001747083
Boundary values, RMSE	0.539597841	0.222369503	0.130686877	0.062929513	0.029214859	0.016299495	0.007563374	0.003655868
Interior gradients, RMSE	0.048413615	0.022677784	0.03298351	0.01516084	0.012860571	0.008869604	0.007413772	0.004109434
Boundary gradients, RMSE	1.231757782	1.301080226	1.18846786	1.163672073	1.222773042	1.190086388	1.187217161	1.178767904
Order (Interior Values, L_∞)		1.45775519	-0.526315843	1.559487319	0.410148873	0.894587369	0.330414957	1.511757764
Order (Boundary values, L_∞)		0.857590718	1.004571908	0.955873352	0.96893666	0.707740587	1.124919552	0.976343639
Order (Interior gradients, L_∞)		0.037468141	-0.9616609	-0.041496532	-0.528868047	-0.02561904	-0.520690643	0.428198162
Order (Boundary gradients, L_∞)		-0.414700383	0.030116185	0.093844482	-0.004528385	-0.095151051	0.000896512	0.057858571
Order (Interior Values, RMSE)		1.372567177	0.480268468	1.335736418	1.033451923	0.824650699	0.884273081	1.187523245
Order (Boundary values, RMSE)		1.278925633	0.766844662	1.054305601	1.107034472	0.841875022	1.107725432	1.048816018
Order (Interior gradients, RMSE)		1.094133116	-0.54046522	1.1213952	0.237395058	0.536012994	0.25866204	0.851268184
Order (Boundary gradients, RMSE)		-0.078991337	0.130607034	0.03041833	-0.071472092	0.039090349	0.003482451	0.010304167



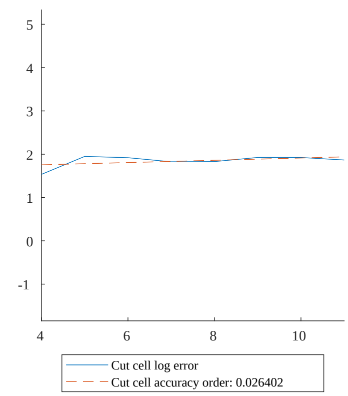
(a) Accuracy of pressure values in non-cut cells



(b) Accuracy of pressure values in cut cells

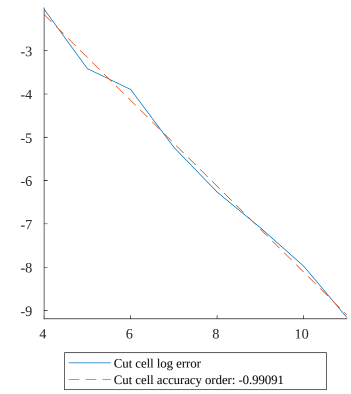


(c) Accuracy of gradient values in non-cut cells

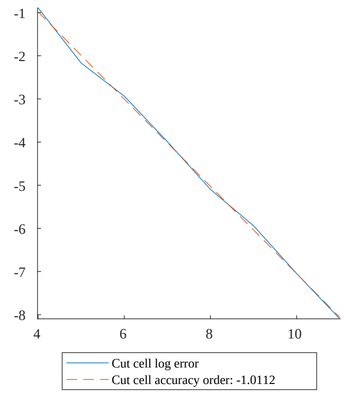


(d) Accuracy of gradient values in cut cells

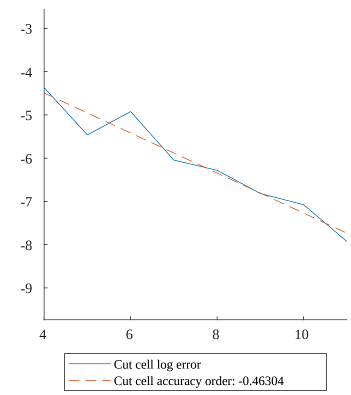
Fig. 12. Convergence of error in the L_∞ with uniform isovalue, parabola function



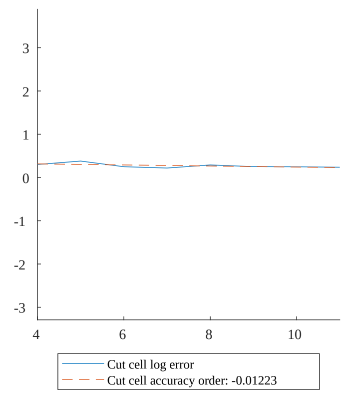
(a) Accuracy of pressure values in non-cut cells



(b) Accuracy of pressure values in cut cells



(c) Accuracy of gradient values in non-cut cells



(d) Accuracy of gradient values in cut cells

Fig. 13. Convergence of error (root-mean-square error) with uniform isovalue, parabola function

This is because as the cells become finer, both the liquid surface and the its iso-surface converge to parallel lines, and it becomes impossible for any line segment that passes though the centroid of the grid edge to intersect the same iso-surface twice.

REFERENCES

[1] Yen Ting Ng, Chohong Min, and Frederic Gibou. An efficient fluid-solid coupling algorithm for single-phase flows. *J. Comput. Physics*, 228:8807–8829, 12 2009.